Efficient Preconditioner of One-sided Space Fractional Diffusion Equation

Xue-lei Lin[1] Michael K. Ng[2] Hai-Wei Sun[3]

Abstract

In this paper, we propose an efficient preconditioner for the linear systems arising from one-sided space fractional diffusion equation with variable coefficients. The shifted Grünwald formula is employed to discretize the one-sided Riemann-Liouville fractional derivative. The resulting linear system is Toeplitz-like which is a summation of a identity matrix and a diagonal-times-nonsymmetric-Toeplitz matrix. A diagonal-times-nonsymmetric-Toeplitz preconditioner is proposed to reduce the condition number of the coefficient matrix, where the diagonal-part comes from the variable diffusion coefficients and the nonsymmetric Toeplitz-part results from the Riemann-Liouville derivative. Theoretically, we show that the condition number of the preconditioned matrix is uniformly bounded by a constant independent of discretization step-sizes under certain assumptions on the coefficient function. When some Krylov subspace method, like the normalized conjugate gradient method, is employed to solve such preconditioned linear system, it converges linearly within a constant iteration number independent of discretization parameters. Numerical results are reported to show the efficiency of the proposed preconditioner and to demonstrate its superiority over the other tested preconditioners.

Keywords: Toeplitz-like matrix; preconditioning; One-sided space-fractional derivative; Variable diffusion coefficients;

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[1]Department of Mathematics, Hong Kong Baptist University, Hong Kong. E-mail: hxuellin@gmail.com

[2]The Corresponding Author. Department of Mathematics, Hong Kong Baptist University, Hong Kong. E-mail: mng@math.hkbu.edu.hk. Research supported in part by HKRGC GRF 12200317, 12306616, 12302715, 12301214.

[3]Department of Mathematics, University of Macau, Macao. E-mail: hsun@umac.mo. This author was supported by research grants MYRG2016-00063-FST from University of Macau and 054/2015/A2 from FDCT of Macao.

1. Introduction

Consider an initial-boundary value problem of space fractional diffusion equation (SFDE) [1, 12, 21]

$$\frac{\partial u(x,t)}{\partial t} = d(x)_a D_x^\alpha u(x,t) + f(x,t), \qquad (x,t) \in (a,b) \times (0,T], \tag{1.1}$$

$$u(a,t) = 0, \qquad t \in (0,T], \tag{1.2}$$

$$u(b,t) = u_R(t), \qquad t \in (0,T], \tag{1.3}$$

$$u(x,0) = \psi(x), \qquad x \in [a,b], \tag{1.4}$$

where $\forall (x,t) \in (a,b)\times[0,T],\ d(x) > \omega_0 > 0$ for some positive constant ω_0, $u(x,t)$ is unknown to be solved, $f(x,t)$ is source term, $\psi(x)$ is initial condition, ${}_aD_x^\alpha u(x,t)$ is the left-sided Riemann-Liouville (RL) derivative defined by [6]

$${}_aD_x^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}} d\xi,$$

$\Gamma(\cdot)$ denotes the gamma function.

Note that the cases, $\alpha = 2$ and $\alpha = 1$ represent the classical diffusion equation and the classical advection equation, respectively, from both of which the linear systems are banded and easy to solve. In this paper, we focus on the case of $\alpha \in (1,2)$. The SFDE (1.1)–(1.4) with $\alpha \in (1,2)$ models a super diffusive flow, in which a cloud of diffusing particles spreads at a faster rate than the classical diffusion model predicts; see [16, 21]. Since closed-form analytical solutions of fractional differential equations are often unavailable especially in the existence of variable coefficients, numerical schemes have been proposed to discretize the SFDE (1.1)–(1.4); see, e.g., [1, 2, 15, 20–23]. Because of the nonlocal property of the RL derivative, these schemes always generate dense linear systems. Thus, direct solver for the linear systems arising from numerical discretization of the SFDE (1.1)–(1.4) requires very expensive computational cost when the grid is dense. This motivates us to develop fast solvers for linear systems arising from the SFDE.

The uniform-grid discretization of the SFDE (1.1)–(1.4) generates Toeplitz-like linear systems which share a coefficient matrix that is a summation of a scalar identity matrix and a diagonal-times-nonsymmetric-Toeplitz matrix. The condition number of the Toeplitz-like coefficient matrix depends on the ratio between temporal and spatial discretization step-sizes, τ/h^α. The condition number is large when τ/h^α is large. Thus, efficient preconditioners are required to reduce the condition number of the coefficient matrix. Although there is no preconditioner proposed exclusively for the one-sided SFDE (1.1)–(1.4), there are several preconditioners proposed for two-sided SFDE, which can be employed to precondition the

Toeplitz-like coefficient matrix arising from one-sided SFDE; see e.g., [4, 7, 9, 19]. Lei and Sun [9] proposed a circulant preconditioner for two-sided SFDE. The circulant preconditioner from [9] is actually the Strang's circulant preconditioner for a Toeplitz matrix deriving from taking averages of the variable coefficients. The theoretical result in [8] is established under the assumption that the diffusion coefficient $d(x)$ is a constant and the assumption that τ/h^{α} is a constant. Pan et al. [19] proposed an approximate inverse preconditioner for two-sided SFDE which is constructed by employing the piece-wise linear interpolation to approximate the inversion of the Toeplitz-like coefficient matrix. The theoretical result in [19] is established under the assumption that τ/h^{α} is bounded by a constant. Jin, Lin and Zhao [7] proposed to use a diagonal-compensated banded truncation of the Toeplitz-like coefficient matrix as a preconditioner for two-sided SFDE, whereas there is no theoretical analysis provided in [7] for analyzing spectra of the preconditioned matrix. Donatelli et al. [4] proposed two structure preserving preconditioners for two-sided SFDE, which are constructed by replacing the dense fractional-order Toeplitz-matrix part in the Toeplitz-like coefficient matrix with the sparse first- and second- order matrices. The theoretical results in [4] are established in distributional sense with the assumption that τ/h^{α} is a constant.

The main objective of this paper is to propose an efficient preconditioner for the Toeplitz-like linear system arising from the one-sided SFDE (1.1)–(1.4) so that preconditioned Krylov subspace methods with the proposed preconditioner converges faster and less dependently on the discretization step-sizes than that with other preconditioners do. In [13], a splitting preconditioner is proposed for preconditioning matrix that is a summation of an identity matrix and a diagonal-times-nonsymmetric-Toeplitz matrix, which is proven to be a good preconditioner in the sense that the resulting preconditioned matrix has an uniformly bounded condition number independent of the discretization step-sizes. Motivated by the spitting preconditioner from [13], in this paper, we propose to use a diagonal-times-nonsymmetric-Toeplitz (DNT) matrix as a preconditioner. In the DNT preconditioner, the diagonal part comes from the diffusion coefficient $d(x)$, and the nonsymmetric Toeplitz part derives from the nonsymmetric discretization of the left-sided RL derivative.

Theoretically, we show that the condition number of the resulting preconditioned matrix is uniformly bounded by a constant independent of discretization step-sizes under some assumptions on the diffusion coefficients $d(x)$. We remark that the proof in this paper is different from that in [13]. Because of the bounded condition number, some Krylov subspace methods, like normalized conjugate gradient method, for the preconditioned linear systems converges within a constant iteration number independent of the discretization step-sizes. Moreover, the matrix-vector multiplication of the preconditioned matrix requires only $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage using the Gohberg-Semencul-type formula and fast Fourier transformations (FFTs) provided that M is the number of spatial-grid points; see [5, 18]. As a result, normalized conjugate gradient method for the preconditioned matrix requires $\mathcal{O}(NM\log M)$ operation and $\mathcal{O}(NM)$ storage, where N is the number of temporal-grid points. Numerical examples are tested to show that the performance of the proposed preconditioner is better than that of other tested preconditioners.

The rest of this paper is organized as follows. In Section 2, we present the Toeplitz-like discretization linear systems of the SFDE (1.1)–(1.4). In Section 3, we introduce the DNT preconditioner for the Toeplitz-like systems and analyze the condition number of the preconditioned matrix. In Section 4, the implementation of the DNT preconditioner is discussed and related experimental results are presented to show the performance of the proposed preconditioner. Finally, some concluding remarks are given in Section 5.

2. Toeplitz-Like Discretization Linear Systems

In this subsection, we present implicit uniform-grid discretization of the FSDE (1.1)–(1.4). and the resulting Toeplitz-like linear systems. For positive integers M and N, let $\tau = T/N$ and $h = (b-a)/(M+1)$. Define the temporal-grid and the spatial-grid respectively by $\{t_n | t_n = n\tau,\ 0 \le n \le N\}$ and $\{x_i | x_i = a + ih,\ 0 \le i \le M+1\}$. Also, we let $\mathbf{x} = (x_1, x_2, ..., x_M)^{\mathrm{T}}$, $d_i = d(x_i)$ for $0 \le i \le M+1$.

For the approximation of the left-sided Riemann-Liouville derivative ${}_aD_x^{\alpha}u$, we refer to the shifted Grünwald formula in [15],

$$ {}_aD_x^{\alpha}u|_{(x,t)=(x_i,t_n)} \approx -\frac{1}{h^{\alpha}}\sum_{j=0}^{i+1} g_{i-j+1}^{(\alpha)} u(x_j, t_n), \qquad 1 \le i \le M, \quad 1 \le n \le N, \tag{2.1}$$

where

$$ g_0^{(\alpha)} = -1, \quad g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha+1}{k+1}\right) g_k^{(\alpha)}, \quad k = 0, 1, 2, ... \tag{2.2}$$

Applying (2.1) to approximating ${}_aD_x^{\alpha}u(x,t)$ and the backward difference to approximating $\frac{\partial u}{\partial t}$, we obtain an implicit difference discretization of SFDE in (1.1)–(1.4) as follows

$$ \tau^{-1}(\mathbf{u}_{n+1} - \mathbf{u}_n) = -h^{-\alpha}\mathbf{D}\mathbf{G}_{\alpha}\mathbf{u}_{n+1} + \mathbf{f}_{n+1}, \quad 0 \le n \le N-1, \tag{2.3}$$

where $\mathbf{u}_0 = \psi(\mathbf{x})$, $\mathbf{u}_n$ is the approximate solution to $u(\mathbf{x}, t_n)$ for $1 \le n \le N$,

$\mathbf{f}_n = f(\mathbf{x}, t_n) + \left(0, 0, ..., 0, -g_0^{(\alpha)} u_R(t_n)/h^\alpha\right)^{\mathrm{T}}$ for $1 \le n \le N$, $\mathbf{D} = \mathrm{diag}(d_1, d_2, ..., d_M)$,

$$\mathbf{G}_\alpha = \begin{bmatrix} g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \cdots & 0 \\ g_2^{(\alpha)} & g_1^{(\alpha)} & g_0^{(\alpha)} & \ddots & \vdots \\ \vdots & g_2^{(\alpha)} & g_1^{(\alpha)} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & g_0^{(\alpha)} \\ g_M^{(\alpha)} & \cdots & \cdots & g_2^{(\alpha)} & g_1^{(\alpha)} \end{bmatrix}.$$

To solve (2.3) is actually equivalent to solve the following linear systems

$$\mathbf{A}\mathbf{u}_n = \mathbf{b}_n, \qquad 1 \le n \le N, \tag{2.4}$$

where

$$\mathbf{A} = \mathbf{I}_M + \eta \mathbf{D}\mathbf{G}_\alpha, \quad \eta = \tau/h^\alpha, \qquad \mathbf{b}_n = \mathbf{u}_{n-1} + \tau \mathbf{f}_n,$$

$\mathbf{I}_k$ denotes $k \times k$ identity matrix, $\mathbf{u}_n$ is unknown to be solved. Here $\mathbf{A}$ is a Toeplitz-like matrix which is a summation of an identity matrix and a diagonal-times-nonsymmetric-Toeplitz matrix. The matrix-vector multiplication of $\mathbf{A}$ requires only $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage; see [18]. Nevertheless, $\mathbf{A}$ is ill-conditioned when η is large; see, e.g., [9, 19]. Thus, a good preconditioner is needed to reduce the condition number of the coefficient matrix in such case.

3. The Preconditioning Method

In this section, we propose a DNT preconditioner for preconditioning $\mathbf{A}$ and analyze condition number of the preconditioned matrix.

For a diagonal matrix $\mathbf{W}$, denote by $\mathtt{mean}(\mathbf{W})$, the mean value of diagonal entries of $\mathbf{W}$. Motivated by the splitting preconditioner in [13], we proposed a DNT preconditioner for $\mathbf{A}$ as follows

$$\mathbf{S} = \sqrt{\mathbf{D}}\mathbf{T}, \tag{3.1}$$

where $\mathbf{T} = \bar{\theta}\mathbf{I}_M + \bar{d}\eta\mathbf{G}_\alpha$ $\bar{d} = \mathtt{mean}(\sqrt{\mathbf{D}})$, $\bar{\theta} = \mathtt{mean}(\mathbf{D}^{-\frac{1}{2}})$.

3.1. Condition Number of the Preconditioned Matrix, $\mathbf{AS}^{-1}$

In this subsection, we estimate the condition number of the preconditioned matrix, $\mathbf{AS}^{-1}$. Before the estimation, we firstly prove the invertibility of the preconditioner $\mathbf{S}$.

Lemma 1.

(i) $g_1^{(\alpha)} > 0,\ \ g_0^{(\alpha)} < g_2^{(\alpha)} < g_3^{(\alpha)} < \cdots \leq 0,\ \sum\limits_{k=0}^{\infty} g_k^{(\alpha)} = 0,$

(ii) $g_k^{(\alpha)} = \frac{-\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}, \quad k \geq 0.$

(iii) $\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}$ *is positive definite.*

Proof: The results (**i**) and (**ii**) can be found directly in [23]. By Gershgorin's circle theorem, (**iii**) follows from (**i**) and (**ii**). □

With Lemma 1, one can easily prove the following proposition.

Proposition 1.

(i) $\mathbf{S}$ *is invertible,*

(ii) $\mathbf{T} + \mathbf{T}^{\mathrm{T}}$ *is positive definite.*

Proof: From (**i**) of Lemma 1, we see that $\mathbf{T}$ is strictly diagonally dominant matrix with positive diagonal entries. Thus, $\mathbf{T}$ is invertible. Moreover, $\sqrt{\mathbf{D}}$ is obviously invertible. Thus, $\mathbf{S}$ is invertible. Again, by (**i**) of Lemma 1, $\mathbf{T} + \mathbf{T}^{\mathrm{T}}$ is strictly diagonally dominant matrix with positive diagonal entries, which is therefore positive definite. The proof is complete. □

Next, we introduce several lemmas and notations, which will come into use in the estimation of the condition number of $\mathbf{A}\mathbf{S}^{-1}$.

Lemma 2. (see [17]) *For any $x \geq 1$, the gamma function holds*

$$\sqrt{\frac{2\pi}{e}}\left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp B_1(x) < \Gamma(x+1) < \sqrt{\frac{2\pi}{e}}\left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp B_2(x),$$

where $e = \exp(1)$ denotes the Euler's number,

$$B_1(x) = \frac{1}{12x} - \frac{1}{12x^2} + \frac{29}{360x^3} - \frac{3}{40x^4} + \frac{17}{252x^5} - \frac{5}{84x^6}, \quad B_2(x) = B_1(x) + \frac{89}{1680x^7}.$$

Lemma 3.

$$\sum_{j=0}^{i} g_j^{(\alpha)} > \frac{c_\alpha}{i^\alpha} > 0, \quad i \geq 1,$$

where $c_\alpha = \frac{\iota_\alpha}{\alpha 3^\alpha}$ with

$$\iota_\alpha = \min\left\{\frac{(3-\alpha)(2-\alpha)(\alpha-1)\alpha 5^{1+\alpha}}{24},\ \frac{1}{\Gamma(-\alpha)}\exp\left(\frac{-1}{1474719}\right)\left(\frac{6e}{5-\alpha}\right)^{1+\alpha}\right\}.$$

Proof: Denote $c_* = \inf\limits_{k\geq 2} |g_k^{(\alpha)}|(k+1)^{1+\alpha}$. Then, $c_* = \min\{c_{*,1}, c_{*,2}\}$ with $c_{*,1} = \min\limits_{2\leq k\leq 4} |g_k^{(\alpha)}|(1+k)^{1+\alpha}$ and $c_{*,2} = \inf\limits_{k\geq 5} |g_k^{(\alpha)}|(1+k)^{1+\alpha}$. Using (2.2), it is easy to check that

$$c_{*,1} = \frac{(3-\alpha)(2-\alpha)(\alpha-1)\alpha 5^{1+\alpha}}{24}. \tag{3.2}$$

Denote $\tilde{k} = k - 1 - \alpha$ for $k \geq 5$. Then, by (**ii**) of Lemma 1 and Lemma 2, we obtain

$$|g_k^{(\alpha)}| = \frac{\Gamma(\tilde{k}+1)}{\Gamma(-\alpha)\Gamma(k+1)} > \left[\frac{\exp(B_1(\tilde{k}) - B_2(k))}{\Gamma(-\alpha)}\right] \left(\frac{\tilde{k}+1}{e}\right)^{\tilde{k}+\frac{1}{2}} \left(\frac{e}{k+1}\right)^{k+\frac{1}{2}}, \quad k \geq 5, \quad (3.3)$$

where $B_1(\cdot)$ and $B_2(\cdot)$ are defined in Lemma 2. By check derivative of $B_1(x)$, it is easy to see that $B_1(x)$ monotonically decreases on the interval $x \in [2, +\infty)$. Therefore,

$$B_1(\tilde{k}) - B_2(k) = B_1(\tilde{k}) - B_1(k) - \frac{89}{1680k^7} \geq \frac{-89}{1680k^7} > \frac{-1}{1474719}, \quad k \geq 5. \quad (3.4)$$

On the other hand,

$$\begin{aligned}
\left(\frac{\tilde{k}+1}{e}\right)^{\tilde{k}+\frac{1}{2}} \left(\frac{e}{k+1}\right)^{k+\frac{1}{2}} &= \frac{e^{1+\alpha}(\tilde{k}+1)^{\tilde{k}+\frac{1}{2}}}{(k+1)^{k+\frac{1}{2}}} \geq \frac{e^{1+\alpha}(\tilde{k}+1)^{\tilde{k}+\frac{1}{2}}}{(\tilde{k}+1)^{k+\frac{1}{2}}} \\
&= \left[\left(\frac{e}{1+k}\right)\left(1 + \frac{1+\alpha}{k-\alpha}\right)\right]^{1+\alpha} \\
&\geq \left[\left(\frac{e}{1+k}\right)\left(1 + \frac{1+\alpha}{5-\alpha}\right)\right]^{1+\alpha} \\
&= \left(\frac{6e}{5-\alpha}\right)^{1+\alpha} \left(\frac{1}{1+k}\right)^{1+\alpha}, \quad k \geq 5. \quad (3.5)
\end{aligned}$$

By (3.3), (3.4) and (3.5), we obtain, $|g_k^{(\alpha)}| > \frac{1}{\Gamma(-\alpha)} \exp\left(\frac{-1}{1474719}\right) \left(\frac{6e}{5-\alpha}\right)^{1+\alpha} \left(\frac{1}{1+k}\right)^{1+\alpha}$ for $k \geq 5$, which implies that $c_{*,2} \geq \frac{1}{\Gamma(-\alpha)} \exp\left(\frac{-1}{1474719}\right) \left(\frac{6e}{5-\alpha}\right)^{1+\alpha}$. This together with (3.2) induces that

$$c_* = \min\{c_{*,1}, c_{*,2}\} \geq \iota_\alpha. \quad (3.6)$$

Moreover, from definition of c^*, we obtain that,

$$|g_k^{(\alpha)}| \geq \frac{c_*}{(1+k)^{1+\alpha}}, \quad \forall k \geq 2. \quad (3.7)$$

By (3.7) and (3.6),

$$\begin{aligned}
\sum_{j=i+1}^{\infty} |g_j^{(\alpha)}| \geq \iota_\alpha \sum_{j=i+1}^{\infty} \frac{1}{(1+j)^{1+\alpha}} &\geq \iota_\alpha \sum_{j=i+1}^{\infty} \int_{j+1}^{j+2} \frac{1}{x^{1+\alpha}} dx \\
&= \frac{\iota_\alpha}{\alpha(i+2)^\alpha} = \left(1 - \frac{2}{2+i}\right)^\alpha \frac{\iota_\alpha}{\alpha i^\alpha} \geq \frac{\iota_\alpha}{\alpha 3^\alpha i^\alpha} = \frac{c_\alpha}{i^\alpha}, \quad i \geq 1.
\end{aligned}$$

Moreover, by (**i**) of Lemma 1, it is easy to see that $\sum\limits_{j=0}^{i} g_j^{(\alpha)} = \sum\limits_{j=i+1}^{\infty} |g_j^{(\alpha)}|$ for $i \geq 1$. The proof is complete. $\square$

Remark: We remark that the expression of the constant ι_α in Lemma 3 can be further simplified. Actually, by numerical illustration, one can find that $\iota_\alpha = \frac{(3-\alpha)(2-\alpha)(\alpha-1)\alpha 5^{1+\alpha}}{24}$.

Let $\gamma \in (1,2)$. Then, for a smooth function $w(x)$, its γ-th order right-sided RL fractional derivative is given by [22]

$$ {}_xD_{x_R}^{\gamma} w(x) := \frac{1}{\Gamma(2-\gamma)} \frac{\partial^2}{\partial x^2} \int_x^{x_R} \frac{w(\xi)}{(\xi - x)^{\gamma-1}} d\xi, $$

where x_R is a real number or $x_R = \infty$.

For a function w, let its Fourier transformation given by $(\mathcal{F}(w))(s) = \int_{-\infty}^{\infty} w(x)\exp(\mathbf{i}sx)dx$, where $\mathbf{i}$ is the imaginary unit.

For $\gamma \in (1,2)$, let

$$ \mathscr{C}^{\gamma+1}(\mathbb{R}) := \left\{ w : \mathbb{R} \to \mathbb{R} \middle| {}_xD_{\infty}^{\gamma+1} w(x) \in L^1(\mathbb{R}),\ \mathcal{F}[{}_xD_{\infty}^{\gamma+1} w(x)] \in L^1(\mathbb{R}) \right\}. $$

For a function $w(x)$ defined on the interval $x \in (a,b)$ which is right-continuous at $x = a$, define a zero extension $\mathcal{Z}(\cdot)$ by

$$ (\mathcal{Z}(w))(x) = \begin{cases} 0, & x \in (-\infty, a-\delta], \\ \text{smooth extension}, & x \in (a-\delta, a], \\ w(x), & x \in (a,b), \\ 0, & x \in [b, \infty), \end{cases} \tag{3.8} $$

where δ is a generic positive constant. The 'smooth extension' mentioned in (3.8) is chosen in the sense that $\mathcal{Z}(w)$ is as smooth as possible. For example, when $w^{(k)}(a+0)$ exists for some nonnegative integer k, then the 'smooth extension' can be a $(2k+1)$-th order Hermite polynomial.

For a function $w(x)$ defined on (a,b) that is right-continuous at $x = a$ and left-continuous at $x = b$, define

$$ (\bar{\mathcal{C}}(w))(x) = \begin{cases} \lim\limits_{x \to a^+} w(x), & x = a, \\ w(x), & x \in (a,b), \\ \lim\limits_{x \to b^-} w(x), & x = b. \end{cases} $$

Denote by $C^k[a,b]$, the set of all functions with continuous k-th order derivative on $[a,b]$ for nonnegative integer k. Especially, $C^0[a,b]$ the set of all continuous functions on $[a,b]$. Recall that $x_i = a + ih$ for $0 \leq i \leq M+1$. For a function, w defined on (a,b), let

$$\mathcal{D}_h^\alpha w(x_i) := -\frac{1}{h^\alpha} \sum_{j=i-1}^{M} g_{j-i+1}^{(\alpha)} w(x_j), \quad 1 \leq i \leq M.$$

Lemma 4. (see [11, 14, 22]) *Let $w(x)$ be a function defined on (a,b). Assume*

(i) *$\bar{\mathcal{C}}(w)$ exists and $\bar{\mathcal{C}}(w) \in C^0[a,b]$,*

(ii) *$\mathcal{Z}(w) \in \mathscr{C}^{\alpha+1}(\mathbb{R})$ with $\lim_{x \to b^-} w(x) = 0$.*

Then,

$$\max_{1 \leq i \leq M} \left| ({}_xD_b^\alpha w)(x_i) - \mathcal{D}_h^\alpha w(x_i) \right| \leq c_w h,$$

where c_w is a positive constant.

For any Hermitian matrices $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{C}^{m \times m}$, denote $\mathbf{H}_1 \prec$ (or $\preceq$)$\mathbf{H}_2$ if $\mathbf{H}_2 - \mathbf{H}_1$ is Hermitian positive definite (or Hermitian positive semi-definite). Especially, we denote $\mathbf{O} \prec$ (or $\preceq$)$\mathbf{H}_1$, when $\mathbf{H}_1$ itself is Hermitian positive definite (or Hermitian positive semi-definite). Similarly, $\mathbf{H}_1 \prec$ (or $\preceq$) $\mathbf{H}_2$ and $\mathbf{O} \prec$ (or $\preceq$)$\mathbf{H}_2$ have the same meanings.

Lemma 5. *Assume*

(i) *$\bar{\mathcal{C}}(d)$ exists and $\bar{\mathcal{C}}(d) \in C^0[a,b]$,*

(ii) *$\mathcal{Z}(\tilde{d}) \in \mathscr{C}^{\alpha+1}(\mathbb{R})$ with $\tilde{d} = d - d_r$ and $d_r = \lim_{x \to b^-} d(x)$,*

(iii) $\inf_{x \in (a,b)} \left[\frac{c_\alpha d(x)}{(x-a)^\alpha} + \frac{c_\alpha d_r}{(b-x)^\alpha} - {}_xD_b^\alpha \tilde{d}(x) \right] > 0$*, with c_α given by Lemma 3.*

Then, there exists a constant positive constant $M_0 > 0$ independent of τ and h such that

$$\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D} \succ \mathbf{O}, \quad \forall M \geq M_0.$$

Proof: Denote $h^{-\alpha}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) = [q_{i,j}]_{i,j=1}^M$. Let $\mathbf{z} = (z_1, z_2, ..., z_M)^{\mathrm{T}}$ be a generic complex vector. By (**i**) of Lemma 1 and nonnegativity of $d(x)$, we have

$$\begin{aligned}
\frac{\mathbf{z}^*(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D})\mathbf{z}}{h^\alpha} = \sum_{i,j=1}^{M} \bar{z}_i q_{i,j} z_j &= \sum_{i=1}^{M} q_{i,i}|z_i|^2 + 2\sum_{i=2}^{M}\sum_{j=1}^{i-1} \mathrm{Re}\left(\bar{z}_i q_{i,j} z_j\right) \\
&\geq \sum_{i=1}^{M} q_{i,i}|z_i|^2 - \sum_{i=2}^{M}\sum_{j=1}^{i-1} |q_{i,j}|(|z_i|^2 + |z_j|^2) \\
&= \sum_{i=1}^{M} |z_i|^2 \sum_{j=1}^{M} q_{i,j} \\
&= \frac{1}{h^\alpha} \sum_{i=1}^{M} |z_i|^2 \left(d_i \sum_{j=1}^{i+1} g_{i-j+1}^{(\alpha)} + \sum_{j=i-1}^{M} g_{i-j+1}^{(\alpha)} d_i \right)
\end{aligned}$$

$$= \sum_{i=1}^{M} |z_i|^2 \left[\frac{d_i}{h^\alpha} \sum_{j=0}^{i} g_j^{(\alpha)} + \frac{d_r}{h^\alpha} \sum_{j=0}^{M+1-i} g_j^{(\alpha)} - \mathcal{D}_h^\alpha (d_i - d_r) \right]. \tag{3.9}$$

By Lemma 3,

$$\frac{d_i}{h^\alpha} \sum_{j=0}^{i} g_j^{(\alpha)} \geq \frac{d_i c_\alpha}{h^\alpha i^\alpha} = \frac{c_\alpha d(x_i)}{(x_i - a)^\alpha}, \quad 1 \leq i \leq M, \tag{3.10}$$

$$\frac{d_r}{h^\alpha} \sum_{j=0}^{M+1-i} g_j^{(\alpha)} \geq \frac{d_r c_\alpha}{h^\alpha (M+1-i)^\alpha} = \frac{c_\alpha d_r}{(b - x_i)^\alpha}, \quad 1 \leq i \leq M. \tag{3.11}$$

Since $\bar{\mathcal{C}}(d) \in C^0[a,b]$, $\bar{\mathcal{C}}(d - d_r) \in C^0[a,b]$ and $\lim\limits_{x \to b^-} \tilde{d}(x) = 0$. Thus, by Lemma 4, there exists a positive constant c_w such that

$$\max_{1 \leq i \leq M} \left| ({}_x D_b^\alpha \tilde{d})(x_i) - \mathcal{D}_h^\alpha (d_i - d_r) \right| \leq c_w h. \tag{3.12}$$

Denote $\nu = \inf\limits_{x \in (a,b)} \left[\frac{c_\alpha d(x)}{(x-a)^\alpha} + \frac{c_\alpha d_r}{(b-x)^\alpha} - {}_x D_b^\alpha \tilde{d}(x) \right]$. By (3.10)–(3.12), it holds that

$$\begin{aligned} \frac{d_i}{h^\alpha} \sum_{j=0}^{i} g_j^{(\alpha)} + \frac{d_r}{h^\alpha} \sum_{j=0}^{M+1-i} g_j^{(\alpha)} - \mathcal{D}_h^\alpha (d_i - d_r) &\geq \frac{c_\alpha d(x_i)}{(x_i - a)^\alpha} + \frac{c_\alpha d_r}{(b - x_i)^\alpha} - ({}_x D_b^\alpha \tilde{d})(x_i) - c_w h \\ &\geq \nu - c_w h, \quad 1 \leq i \leq M. \end{aligned}$$

By (**iii**), $\nu > 0$. Take $M_0 = \max\{\lceil 2c_d(b-a)/\nu \rceil, 1\}$. Then, we see that

$$\min_{1 \leq i \leq M} \left[\frac{d_i}{h^\alpha} \sum_{j=0}^{i} g_j^{(\alpha)} + \frac{d_r}{h^\alpha} \sum_{j=0}^{M+1-i} g_j^{(\alpha)} - \mathcal{D}_h^\alpha (d_i - d_r) \right] > 0, \quad M \geq M_0. \tag{3.13}$$

By (3.9) and (3.13), $\mathbf{z}^*(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D})\mathbf{z} > 0$ for $M \geq M_0$. Thus, $\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D} \succ \mathbf{O}$ for $M \geq M_0$. □

Proposition 2. *For positive numbers θ_i, η_i, $1 \leq i \leq m$, it holds that*

$$\min_{1 \leq i \leq m} \frac{\theta_i}{\eta_i} \leq \left(\sum_{i=1}^{m} \eta_i \right)^{-1} \left(\sum_{i=1}^{m} \theta_i \right) \leq \max_{1 \leq i \leq m} \frac{\theta_i}{\eta_i}.$$

Proof: Denote $a_1 = \min\limits_{1 \leq i \leq m} \theta_i/\eta_i$ and $a_2 = \max\limits_{1 \leq i \leq m} \theta_i/\eta_i$. The result follows from

$$a_1 = \left(\sum_{i=1}^{m} \eta_i \right)^{-1} \left(\sum_{i=1}^{m} a_1 \eta_i \right) \leq \left(\sum_{i=1}^{m} \eta_i \right)^{-1} \left(\sum_{i=1}^{m} \theta_i \right) \leq \left(\sum_{i=1}^{m} \eta_i \right)^{-1} \left(\sum_{i=1}^{m} a_2 \eta_i \right) = a_2.$$

□

For $\mathbf{C} \in \mathbb{R}^{m\times n}$, let $\Sigma(\mathbf{C})$ denote the set of singular values of $\mathbf{C}$. Also denote $\Sigma^2(\mathbf{C}) = \{\sigma^2 | \sigma \in \Sigma(\mathbf{C})\}$. For any invertible matrix, $\mathbf{C} \in \mathbb{R}^{m\times m}$, define its condition number as $\text{cond}(\mathbf{C}) := ||\mathbf{C}||_2 ||\mathbf{C}^{-1}||_2$.

Theorem 6. *Assume the assumptions in Lemma 5 hold. Then, there exists an $M_0 > 0$ independent of τ and h such that*

$$\sup_{N\geq 1}\sup_{M\geq M_0} \text{cond}(\mathbf{A}\mathbf{S}^{-1}) \leq \sqrt{\hat{s}/\check{s}},$$

where the positive constants $\hat{s}$, $\check{s}$ are as follows

$$\hat{s} = \max\left\{\frac{\mu}{\omega_0}, \sqrt{\frac{\mu}{\omega_0}}\right\}, \quad \check{s} = \min\left\{\frac{\omega_0}{\mu}, \sqrt{\frac{\omega_0}{\mu}}\right\},$$

with the constants M_0 given by Lemma 5 and $\mu = \sup\limits_{x\in(a,b)} d(x)$

Proof: By straightforward calculation,

$$\begin{aligned}
\mathbf{A}^{\mathrm{T}}\mathbf{A} &= \mathbf{I}_M + \eta(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}^2\mathbf{G}_\alpha,\\
\mathbf{S}^{\mathrm{T}}\mathbf{S} &= \bar{\theta}^2\mathbf{D} + \eta\bar{\theta}\bar{d}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\bar{d}^2\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha.
\end{aligned}$$

By Lemma 5 and the fact that $\mathbf{O} \prec \omega_0\mathbf{I}_M \preceq \mathbf{D} \preceq \mu\mathbf{I}_M$, we obtain that for $M \geq M_0$,

$$\begin{aligned}
\mathbf{O} &\prec \mathbf{I}_M + \eta(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\omega_0\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha\\
&\prec \mathbf{A}^{\mathrm{T}}\mathbf{A} \prec \mathbf{I}_M + \eta(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\mu\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\mathbf{O} &\prec \mu^{-1}\omega_0\mathbf{I}_M + \eta\sqrt{\mu^{-1}\omega_0}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\omega_0\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha\\
&\prec \mathbf{S}^{\mathrm{T}}\mathbf{S} \prec \mu\omega_0^{-1}\mathbf{I}_M + \eta\sqrt{\mu\omega_0^{-1}}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\mu\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha.
\end{aligned} \tag{3.15}$$

For any non-zero vector $\mathbf{z} \in \mathbb{R}^{M\times 1}$, let $\mathbf{w} = \mathbf{S}^{-1}\mathbf{z}$. Then,

$$\frac{\mathbf{z}^{\mathrm{T}}(\mathbf{A}\mathbf{S}^{-1})^{\mathrm{T}}(\mathbf{A}\mathbf{S}^{-1})\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}} = \frac{\mathbf{w}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{w}}{\mathbf{w}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{w}}. \tag{3.16}$$

By (3.14), (3.15) and Proposition 2,

$$\check{s} = \min\left\{\frac{\omega_0}{\mu}, \sqrt{\frac{\omega_0}{\mu}}\right\} \leq \frac{\mathbf{w}^{\mathrm{T}}[\mathbf{I}_M + \eta(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\omega_0\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha]\mathbf{w}}{\mathbf{w}^{\mathrm{T}}[\mu\omega_0^{-1}\mathbf{I}_M + \eta\sqrt{\mu\omega_0^{-1}}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\mu\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha]\mathbf{w}}$$

$$
\begin{aligned}
&\leq \frac{\mathbf{w}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{w}}{\mathbf{w}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{w}} \\
&\leq \frac{\mathbf{w}^{\mathrm{T}}[\mathbf{I}_M + \eta(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\mu\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha]\mathbf{w}}{\mathbf{w}^{\mathrm{T}}[\mu^{-1}\omega_0\mathbf{I}_M + \eta\sqrt{\mu^{-1}\omega_0}(\mathbf{D}\mathbf{G}_\alpha + \mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}) + \eta^2\omega_0\mathbf{G}_\alpha^{\mathrm{T}}\mathbf{D}\mathbf{G}_\alpha]\mathbf{w}} \\
&\leq \max\left\{\frac{\mu}{\omega_0}, \sqrt{\frac{\mu}{\omega_0}}\right\} = \hat{s}, \quad M \geq M_0,
\end{aligned}
$$

which together with (3.16) implies that $\Sigma^2(\mathbf{A}\mathbf{S}^{-1}) \subset [\check{s}, \hat{s}]$ for $M \geq M_0$. Hence,

$$
\sup_{N\geq 1}\sup_{M\geq M_0} \operatorname{cond}(\mathbf{A}\mathbf{S}^{-1}) \leq \sqrt{\hat{\mathrm{s}}/\check{\mathrm{s}}},
$$

which completes the proof. □

Remark: Theorem 6 shows that when the coefficient function satisfies the related assumptions, then the condition number of the preconditioned matrix, $\mathbf{A}\mathbf{S}^{-1}$ will be finally bounded by a discretization-parameter independent constant as M getting larger. On the other hand, as M getting larger, the ill-conditioned part, $\mathbf{G}_\alpha$ in $\mathbf{A}$ gradually dominates, which leads to the matrix $\mathbf{A}$ becoming more ill-conditioned. That means, our proposed DNT preconditioner significantly reduces the condition number so that some iterative solvers, like the normalized conjugate gradient method, for the preconditioned linear system converges within a constant iteration number.

4. Numerical Experiments

In this section, we firstly discuss the implementation of the DNT preconditioner and extend it to two-dimensional SFDE. Then, several examples are tested to demonstrate the efficiency of the proposed DNT preconditioner and compare it with that of other preconditioners.

4.1. Implementation of One-Dimensional Case

When using iterative solver to solve the preconditioned linear system, it requires to compute some matrix-vector multiplications associated with the matrix $\mathbf{A}\mathbf{S}^{-1}$. Moreover, as mentioned above, the Toeplitz-like matrix $\mathbf{A}$ has fast matrix-vector multiplication. Moreover, $\mathbf{S}^{-1} = \mathbf{T}^{-1}\mathbf{D}^{-\frac{1}{2}}$ and $\mathbf{D}^{-\frac{1}{2}}$ is a diagonal matrix. Thus, we only need to discuss the implementation of $\mathbf{T}^{-1}\mathbf{z}$ for some given $\mathbf{z}$. Similarly to the implementation of splitting preconditioner in [13], we employ the Gohberg-Semencul-type formula [5] to implement $\mathbf{T}^{-1}\mathbf{z}$.

Let $\mathbf{v} = (v_1, v_2, ..., v_M)^{\mathrm{T}}$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_M)^{\mathrm{T}}$ be solutions of following linear systems

$$
\mathbf{T}\mathbf{v} = \mathbf{e}_1 \equiv (1, 0, 0, ..., 0)^{\mathrm{T}} \in \mathbb{R}^{M\times 1}, \qquad \mathbf{T}\tilde{\mathbf{v}} = \mathbf{e}_M \equiv (0, 0, ..., 0, 1)^{\mathrm{T}} \in \mathbb{R}^{M\times 1}. \tag{4.1}
$$

According to the Gohberg-Semencul-type formula [5], $\mathbf{T}^{-1}$ can be expressed as follows

$$\mathbf{T}^{-1} = \frac{1}{2v_1}(\mathbf{S}_1\mathbf{C}_1 - \mathbf{S}_2\mathbf{C}_2), \tag{4.2}$$

where $\mathbf{S}_1$, $\mathbf{S}_2$ are skew-circulant matrices with $\mathbf{v}$, $\bar{\mathbf{v}} = (-\tilde{v}_M, \tilde{v}_1, ..., \tilde{v}_{M-1})^{\mathrm{T}}$ as their first columns, respectively; $\mathbf{C}_1$, $\mathbf{C}_2$ are circulant matrices with $\hat{\mathbf{v}} = (\tilde{v}_M, \tilde{v}_1, ..., \tilde{v}_{M-1})^{\mathrm{T}}$, $\mathbf{v}$ as their first columns, respectively. From (4.1), we see that v_1 is the first diagonal entry of $\mathbf{T}^{-1}$. From (**ii**) of Proposition 1, $\mathbf{T} + \mathbf{T}^{\mathrm{T}}$ is positive definite. Thus,

$$v_1 = \mathbf{e}_1^{\mathrm{T}}\mathbf{T}^{-1}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_1^{\mathrm{T}}(\mathbf{T}^{-1} + \mathbf{T}^{-\mathrm{T}})\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_1^{\mathrm{T}}\mathbf{T}^{-1}(\mathbf{T} + \mathbf{T}^{\mathrm{T}})\mathbf{T}^{-\mathrm{T}}\mathbf{e}_1 > 0,$$

which means (4.2) is applicable. Moreover, the Toeplitz linear systems in (4.1) can be efficiently solved by many existing solvers; see, e.g., [3, 18]. With (4.2), the matrix-vector multiplication $\mathbf{T}^{-1}\mathbf{z}$ requires $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage for any given $\mathbf{z}$. Thus, we conclude that the matrix-vector multiplication of the preconditioned matrix, $\mathbf{A}\mathbf{S}^{-1}\mathbf{z}$ requires $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage for any given $\mathbf{z}$.

4.2. Extension to Two-Dimensional Case

In this subsection, we extend the DNT preconditioner to two-dimensional FSDE and discuss its implementation. Consider a two-dimensional initial-boundary value problem of space-fractional diffusion equation [20]:

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\ {}_{a_1}D_x^{\alpha}u(x,y,t) + e(x,y)\ {}_{a_2}D_y^{\beta}u(x,y,t) + f(x,y,t),\ (x,y,t) \in \Omega \times (0,T], \tag{4.3}$$

$$u(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{4.4}$$

$$u(x,y,0) = \psi(x,y), \quad (x,y) \in \bar{\Omega}, \tag{4.5}$$

where $\Omega = (a_1, b_1) \times (a_2, b_2)$, $\partial\Omega$ denotes boundary of Ω, $d(x,y)$ and $e(x,y)$ are known functions that are larger than a positive constant, $f(x,y,t)$ is source term, $\psi(x,y)$ is initial condition, $\alpha, \beta \in (1,2)$.

For positive integers M and N, let $\tau = T/N$, $h_x = (b_1 - a_1)/(M+1)$, $h_y = (b_2 - b_1)/(M+1)$. Define the temporal girds, spatial grids in x-direction and spatial grids in y-direction, by

$$\{t_n = n\tau | 0 \le n \le N\},\ \{x_i = a_1 + ih_x | 0 \le i \le M+1\},\ \{y_j = b_1 + jh_y | 0 \le j \le M+1\},$$

respectively. Then, the vectors consisting of spatial-grid points with x-dominant ordering and

y-dominant ordering are defined respectively by

$$\mathscr{P}_{x,M} = (P_{1,1}, P_{2,1}, ..., P_{M,1}, P_{1,2}, P_{2,2}, ..., P_{M,2}, ..., P_{1,M}, P_{2,M}, ..., P_{M,M})^{\mathrm{T}}, \tag{4.6}$$

$$\mathscr{P}_{y,M} = (P_{1,1}, P_{1,2}, ..., P_{1,M}, P_{2,1}, P_{2,2}, ..., P_{2,M}, ..., P_{M,1}, P_{M,2}, ..., P_{M,M})^{\mathrm{T}}, \tag{4.7}$$

where $P_{i,j}$ denotes the point (x_i, y_j) for $0 \leq i, j \leq M+1$. Then, using (2.1) and the backward difference approximation to $\frac{\partial u}{\partial t}$, we obtain two-level Toeplitz-like linear systems as follows

$$\mathbf{A}\mathbf{u}_n = \mathbf{b}_n, \qquad 1 \leq n \leq N, \tag{4.8}$$

where $\mathbf{b}_n = \tau \mathbf{f}^n + \mathbf{u}^{n-1}$, $\mathbf{f}^n = f(\mathscr{P}_{x,M}, t_n)$, the solution $\mathbf{u}_n$ is an approximation to $u(\mathscr{P}_{x,M}, t_n)$, $\mathbf{A} = \mathbf{I}_{M^2} + \eta_x \mathbf{D}(\mathbf{I}_M \otimes \mathbf{G}_\alpha) + \eta_y \mathbf{E}(\mathbf{G}_\beta \otimes \mathbf{I}_M)$, $\eta_x = \tau / h_x^\alpha$, $\eta_y = \tau / h_y^\beta$, "$\otimes$" denotes the Kronecker product, $\mathbf{D} = \operatorname{diag}(d(\mathscr{P}_{x,M}))$, $\mathbf{E} = \operatorname{diag}(e(\mathscr{P}_{x,M}))$.

The two-level DNT preconditioner for the two-level Toeplitz-like matrix $\mathbf{A}$ is defined as follows

$$\mathbf{S} = \mathbf{W}\mathbf{T},$$

where $\mathbf{W} = \sqrt{\mathbf{D} + \mathbf{E}}$, $\mathbf{T} = \bar{\theta}\mathbf{I}_{M^2} + \eta_x \bar{d} \mathbf{I}_M \otimes \mathbf{G}_\alpha + \eta_y \bar{e} \mathbf{G}_\beta \otimes \mathbf{I}_M$, $\bar{\theta} = \mathtt{mean}(\mathbf{W}^{-1})$, $\bar{d} = \mathtt{mean}(\mathbf{D}\mathbf{W}^{-1})$, $\bar{d} = \mathtt{mean}(\mathbf{E}\mathbf{W}^{-1})$.

Notice that $\mathbf{A}$ is a two-level Toeplitz-like matrix, matrix-vector multiplication of which can be fast done with $\mathcal{O}(M^2 \log M)$ operation and $\mathcal{O}(M^2)$ storage using properties of Kronecker product and FFTs; see [18]. Moreover, $\mathbf{S}^{-1} = \mathbf{T}^{-1}\mathbf{W}^{-1}$ and $\mathbf{W}^{-1}$ is a diagonal matrix. For the implementation of $\mathbf{T}^{-1}\mathbf{z}$, we employ the same multigrid method as it is used in the implementation of the two-dimensional splitting preconditioner in [13]. As a result, matrix-vector multiplication of the preconditioned matrix only requires $\mathcal{O}(M^2 \log M)$ operation and $\mathcal{O}(M^2)$ storage; see [13].

4.3. Numerical Examples

In this subsection, we use several examples to test the performance of the proposed DNT preconditioner and compare it with the performance of other preconditioners. The generalized minimum residual (GMRES) method is applied to solve the preconditioned Toeplitz-like linear systems. All numerical experiments are performed via MATLAB R2015a on a PC with the configuration: Intel(R) Core(TM) i7-4720HQ CPU 2.60 GHz and 8 GB RAM.

Other testing preconditioners for (2.4) and (4.8) are listed as follows. The circulant preconditioner [9] and the multilevel circulant preconditioner [10] can be used to precondition (2.4) and (4.8), respectively. For convenience, we use $\mathbf{C}$ to denote (multilevel) circulant preconditioner. FFTs are used to compute the corresponding preconditioned matrix-vector multiplication. Denote by $\mathbf{P}(k)$, the approximate inverse preconditioner [19] with k interpolating points for

(2.4) while with k interpolating points in both x and y direction, respectively for (4.8); see [19]. FFTs are used to compute the corresponding preconditioned matrix-vector multiplication. Denote by $\mathbf{B}(k)$, the banded preconditioner of bandwidth k for $\mathbf{A}$ from (2.4) or (4.8); see [7]. Banded solvers are used to compute the corresponding preconditioned matrix-vector multiplication. Also, denote by $\mathbf{S}_1$ and $\mathbf{S}_2$, the two structure preserving preconditioners proposed in [4], for which the one dimensional implementation is already discussed in [4]. In two-dimensional case, $\mathbf{S}_1$ and $\mathbf{S}_2$ are defined as

$$\mathbf{S}_1 = \mathbf{I}_{M^2} + \eta_x\sigma_\alpha\mathbf{D}(\mathbf{I}_M \otimes \mathbf{L}_M) + \eta_y\sigma_\beta\mathbf{E}(\mathbf{L}_M \otimes \mathbf{I}_M), \quad \mathbf{L}_M = \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{M\times M}$$
$$\mathbf{S}_2 = \mathbf{I}_{M^2} + 2\eta_x\sigma_\alpha\mathbf{D}(\mathbf{I}_M \otimes \hat{\mathbf{L}}_M) + 2\eta_y\sigma_\beta\mathbf{E}(\hat{\mathbf{L}}_M \otimes \mathbf{I}_M), \quad \hat{\mathbf{L}}_M = \mathbf{L}_M + \mathbf{L}_M^{\mathrm{T}}.$$

The $\mathbf{S}_1$ and $\mathbf{S}_2$ defined above can be implemented using the same multigrid method as the one used for implementation of two-level DNT preconditioner.

We employ the preconditioned GMRES method with different preconditioners to solve linear systems (2.4) and linear systems (4.8). Also, denote by GMRES-**DNT**, GMRES-**C**, GMRES-$\mathbf{P}(k)$, GMRES-$\mathbf{B}(k)$, GMRES-$\mathbf{S}_1$, GMRES-$\mathbf{S}_2$, the preconditioned GMRES method with preconditioners, DNT, $\mathbf{C}$, $\mathbf{P}(k)$, $\mathbf{B}(k)$, $\mathbf{S}_1$, $\mathbf{S}_2$, respectively. Especially, we denote by GMRES-**I**, the GMRES iteration without preconditioner. For all of these methods, we set $\mathbf{u}_{n-1}$ as initial guess and set $\frac{||\mathbf{r}_k||_2}{||\mathbf{r}_0||_2} \leq$1e-7 as stopping criterion, where $\mathbf{r}_k$ denotes residual vector in k-th iteration. All preconditioned GMRES methods tested here are restarted versions with a restarting number, 300.

Define the relative error

$$\mathrm{E}_{N,M} = \frac{||\mathbf{u} - \tilde{\mathbf{u}}||_\infty}{||\mathbf{u}||_\infty},$$

where $\mathbf{u}$ and $\tilde{\mathbf{u}}$ denote the exact solution of the SFDE (1.1)–(1.4) and the iterative solution the Toeplitz-like linear systems (2.4) or (4.8), respectively. Denote by 'CPU', the running time, units of which are 's' for second and 'h' for hour, respectively. Denote by aeb, the number $a \times 10^b$. Notice that there are N linear systems in (2.4) or (4.8). We denote by 'iter', the average iteration number of GMRES solver for the N linear systems.

Example 1. Consider the SFDE (1.1)–(1.4) with

$$u(x,t) = t^2x^4(2-x)^4, \quad d(x) = \exp(12 + \sin(20x)\cos(20x)), \quad T = 1, \quad [a,b] = [0,2]$$
$$f(x,t) = 2tx^4(2-x)^4 - d(x)t^2\sum_{i=5}^{9}\frac{q_i\Gamma(i)x^{i-1-\alpha}}{\Gamma(i-\alpha)}, \ q_5 = 16, \ q_6 = -32, \ q_7 = 24, \ q_8 = -8, \ q_9 = 1.$$

We solve Example 1 using preconditioned GMRES method with different preconditioners. The corresponding results are listed in Tables 1–4. Since $\mathrm{E}_{N,M}$ resulting from those solvers tested are always small and almost the same, for which results of $\mathrm{E}_{N,M}$ are not listed Tables 1–4. We remark that the other values of k in GMRES-$\mathbf{P}(k)$ and GMRES-$\mathbf{B}(k)$ do not make too much difference on CPU cost, for which results of GMRES-$\mathbf{P}(k)$ and GMRES-$\mathbf{B}(k)$ with other values of k are not listed in Tables 1–4. Tables 1–4 show that the performance of the DNT preconditioner is better than that of other preconditioners tested in terms of both iterations and computational times.

To further illustrate the effectiveness of the DNT preconditioner, we also list the condition numbers of the coefficient matrix and the preconditioned matrix by DNT preconditioner at the final time level for $N = 1$ and different values of $\eta = \tau/h^\alpha$ in Table 5. In this example, $\tau = 1/N$ and $h = 2/(M+1)$ are the temporal and spatial discretization step-sizes, respectively, and thus $\eta = (M+1)^\alpha/(2^\alpha N)$. Comparing the condition number of $\mathbf{A}$ in Tables 5, we see that the condition number of the coefficient matrix is large when η is large. On the other hand, the condition number of the preconditioned matrix by DNT preconditioner is always small and almost unchanged as η changes. That means the condition number of the preconditioned matrix is independent of τ and h, which is in accordance with the theoretical results. In addition, if η goes to zero, then the coefficient matrix $\mathbf{A}$ tends to identity matrix and thus is well-conditioned. Thus, $\mathbf{A}$ with small η does not need a preconditioner.

Table 1: Results of different solvers when $N = 2^7$ for Example 1.

		GMRES-**DNT**		GMRES-$\mathbf{B}(15)$		GMRES-$\mathbf{S}_1$		GMRES-$\mathbf{S}_2$	
α	$M+1$	iter	CPU	iter	CPU	iter	CPU	iter	CPU
1.2	2^{12}	6.8	1.79s	11.0	3.35s	>4000	>2h	148.9	153.04s
	2^{13}	6.8	5.12s	12.4	6.79s	>6000	>4h	199.2	415.77s
	2^{14}	6.8	10.15s	13.8	14.29s	>8000	>8h	270.5	1133.30s
1.5	2^{12}	6.8	1.70s	30.2	9.87s	>4000	>2h	56.2	29.12s
	2^{13}	6.8	5.12s	39.6	24.44s	>6000	>4h	67.3	63.33s
	2^{14}	6.8	10.18s	51.2	60.84s	>8000	>8h	80.0	139.37s
1.8	2^{12}	6.8	1.76s	43.7	16.93s	>4000	>2h	16.1	5.47s
	2^{13}	6.8	5.06s	72.0	64.79s	>6000	>4h	17.6	10.54s
	2^{14}	6.8	10.20s	117.2	237.21s	>8000	>8h	19.2	20.98s

Table 2: Results of different solvers when $N = 2^7$ for Example 1.

		GMRES-$\mathbf{C}$		GMRES-$\mathbf{P}(5)$		GMRES-$\mathbf{I}$	
α	$M+1$	iter	CPU	iter	CPU	iter	CPU
1.2	2^{12}	11.7	3.18s	27.8	12.35s	>4000	>2h
	2^{13}	11.8	6.50s	27.3	19.95s	>8000	>8h
	2^{14}	11.8	11.54s	27.4	33.42s	>16000	>64h
1.5	2^{12}	12.1	3.11s	24.8	10.19s	>4000	>2h
	2^{13}	12.1	6.46s	24.5	16.99s	>8000	>8h
	2^{14}	12.1	11.88s	24.1	28.01s	>16000	>64h
1.8	2^{12}	12.0	2.94s	22.0	8.45s	>4000	>2h
	2^{13}	12.1	6.43s	21.8	14.27s	>8000	>8h
	2^{14}	12.1	11.77s	21.5	23.95s	>16000	>64h

Table 3: Results of different solvers when $M+1=2^{13}$ for Example 1.

α	N	GMRES-**DNT**		GMRES-**B**(15)		GMRES-$\mathbf{S}_1$		GMRES-$\mathbf{S}_2$	
		iter	CPU	iter	CPU	iter	CPU	iter	CPU
1.2	2^8	6.4	9.49s	11.7	12.43s	>3000	>5h	197.6	851.95s
	2^9	6.2	18.48s	11.0	23.73s	>3000	>10h	196.7	1705.37s
	2^{10}	5.8	35.32s	10.5	45.64s	>3000	>20h	194.6	3501.63s
1.5	2^8	6.4	9.48s	37.7	45.91s	>3000	>5h	64.6	120.43s
	2^9	6.2	18.44s	35.9	86.04s	>3000	>10h	62.4	226.75s
	2^{10}	5.8	35.55s	34.1	161.99s	>3000	>20h	60.6	429.88s
1.8	2^8	6.4	9.55s	70.0	122.95s	>3000	>5h	16.8	19.86s
	2^9	6.2	18.62s	67.9	233.33s	>3000	>10h	15.9	38.36s
	2^{10}	5.8	35.23s	65.5	312.50s	>3000	>20h	15.0	72.28s

Table 4: Results of different solvers when $M=2^{13}$ for Example 1.

α	N	GMRES-**C**		GMRES-**P**(5)		GMRES-**I**	
		iter	CPU	iter	CPU	iter	CPU
1.2	2^8	11.4	12.51s	22.8	31.52s	>8000	>6h
	2^9	10.8	24.02s	19.2	49.83s	>8000	>12h
	2^{10}	10.4	45.44s	16.6	80.95s	>8000	>24h
1.5	2^8	11.5	12.30s	20.8	21.26s	>8000	>6h
	2^9	11.1	23.69s	18.0	44.87s	>8000	>12h
	2^{10}	10.6	44.96s	15.7	74.26s	>8000	>24h
1.8	2^8	11.5	12.25s	18.7	22.94s	>8000	>6h
	2^9	11.1	23.71s	16.7	38.98s	>8000	>12h
	2^{10}	10.6	45.29s	14.8	65.54s	>8000	>24h

Table 5: Condition numbers of **A** and the preconditioned matrix $\mathbf{AT}^{-1}$ by Toeplitz preconditioner in Example 1 for different values of η.

α	1.2			1.5			1.8		
$\eta=\frac{(M+1)^\alpha}{2^\alpha N}$	1.78e3	4.10e3	9.41e3	1.16e4	3.28e4	9.27e4	7.53e4	2.62e5	9.13e5
A	7.10e3	1.64e4	3.79e4	4.00e4	1.14e5	3.24e5	2.35e5	8.27e5	2.89e6
$\mathbf{AT}^{-1}$	3.31	3.31	3.32	3.31	3.31	3.32	3.31	3.32	3.32

Example 2. Consider the two-dimensional space-fractional diffusion equation (4.3)–(4.5) with

$$u(x,y,t)=t^2x^4(2-x)^4y^4(2-y)^4,\quad [a_1,b_1]=[0,2],\quad [a_2,b_2]=[0,2],\quad T=1,$$
$$d(x,y)=3+x^2+y^2,\quad e(x,y)=3+\sin((4+x)\pi)+\sin((4+y)\pi),$$
$$f(x,y,t)=2tx^4(2-x)^4y^4(2-y)^4-t^2y^4(2-y)^4d(x,y)\sum_{i=5}^{9}\frac{q_i\Gamma(i)x^{i-1-\alpha}}{\Gamma(i-\alpha)}-$$

$$t^2x^4(2-x)^4e(x,y)\sum_{i=5}^{9}\frac{q_i\Gamma(i)y^{i-1-\beta}}{\Gamma(i-\beta)},$$

$$q_5=16,\quad q_6=-32,\quad q_7=24,\quad q_8=-8,\quad q_9=1.$$

We solve Example 2 by the right-preconditioned GMRES method with different preconditioners. The corresponding numerical results are listed in Tables 6–7. Since $\mathrm{E}_{M,N}$ of the different solvers are all small and almost the same, results of $\mathrm{E}_{M,N}$ are not listed in the tables. From Tables 6–7, we see that the performance of the proposed DNT preconditioner is generally better than that of other solvers in terms of both iterations and computational times.

Table 6: Results of different solvers when $N=2^4$ for Example 2.

		GMRES-**DNT**		GMRES-**P**(5)		GMRES-**B**(15)		GMRES-**C**	
(α,β)	$M+1$	iter	CPU	iter	CPU	iter	CPU	iter	CPU
(1.1,1.6)	2^8	15.6	19.23s	17.9	30.21s	110.5	92.68s	29.4	12.98s
	2^9	15.9	93.04s	21.6	195.35s	>200	>1h	35.2	120.13s
	2^{10}	15.7	411.38s	26.1	917.88s	>200	>2h	42.4	727.89s
(1.1,1.9)	2^8	17.4	20.58s	28.8	47.30s	174.0	198.44s	44.4	21.87s
	2^9	17.1	101.79s	38.0	340.70s	>400	>2h	57.5	250.78s
	2^{10}	18.9	500.52s	50.1	1983.65s	>400	>4h	74.6	1869.09s
(1.6,1.6)	2^8	13.9	16.80s	18.6	30.88s	111.7	93.89s	28.7	12.56s
	2^9	13.9	83.05s	21.8	188.37s	>200	>1h	33.0	117.11s
	2^{10}	13.8	357.82s	25.5	850.64s	>200	>2h	37.8	614.29s
(1.9,1.9)	2^8	13.9	16.95s	24.8	38.74s	177.8	209.83s	33.1	15.11s
	2^9	13.6	81.57s	30.2	256.72s	>400	>2h	39.1	139.25s
	2^{10}	13.3	342.02s	37.1	1287.00s	>400	>4h	46.2	828.53s

Table 7: Results of different solvers when $N=2^4$ for Example 2.

		GMRES-$\mathbf{S_1}$		GMRES-$\mathbf{S}_2$		GMRES-**I**	
(α,β)	$M+1$	iter	CPU	iter	CPU	iter	CPU
(1.1,1.6)	2^8	>1000	>1h	43.4	55.33s	1170.1	1945.39s
	2^9	>1000	>2h	49.8	342.97s	>1000	>1h
	2^{10}	>1000	>4h	55.4	1828.28s	>1000	>2h
(1.1,1.9)	2^8	>2000	>2h	45.4	58.10s	>2000	>1h
	2^9	>2000	>4h	51.0	358.81s	>2000	>2h
	2^{10}	>2000	>8h	55.9	1851.11s	>2000	>4h
(1.6,1.6)	2^8	>1000	>1h	20.3	25.72s	>702.9	1145.26s
	2^9	>1000	>2h	21.1	125.33s	>2000	>2h
	2^{10}	>1000	>4h	21.9	576.54s	>2000	>4h
(1.9,1.9)	2^8	>2000	>2h	17.1	22.02s	>873.9	>1447.43s
	2^9	>2000	>4h	17.8	105.32s	>2000	>2h
	2^{10}	>2000	>8h	18.3	472.90s	>2000	>4h

5. Concluding Remark

In this paper, we have considered linear systems arising from discretization of time dependent one-sided space fractional diffusion equation with variable coefficient. The coefficient matrix is a summation of an identity matrix and diagonal-times-nonsymmetric-Toeplitz matrices. The main contribution of this paper is to propose a DNT preconditioner for such linear systems so that preconditioned Krylov subspace methods with the DNT preconditioner converges faster than that with other preconditioners do. Theoretically, we have shown that the condition number of the preconditioned matrix by the proposed DNT preconditioner is bounded by a constant independent of discretization step-sizes under certain assumptions presented in Theorem 6 that are related to diffusion coefficient function. Numerical results reported have shown the high performance of the proposed DNT preconditioner. We will consider extend the DNT preconditioner to SFDE with two-sided fractional derivatives weighted by different coefficient functions.

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