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Stability and Convergence Analysis of Finite Difference Schemes for Time-Dependent Space-Fractional Diffusion Equations with Variable Diffusion Coefficients

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Abstract In this paper, we study and analyze Crank–Nicolson temporal discretization with high-order spatial difference schemes for time-dependent Riesz space-fractional diffusion equations with variable diffusion coefficients. To the best of our knowledge, there is no stability and convergence analysis for temporally 2nd-order or spatially jth-order ($j \geq 3$) difference schemes for such equations with variable coefficients. We prove under mild assumptions on diffusion coefficients and spatial discretization schemes that the resulting discretized systems are unconditionally stable and convergent with respect to discrete ℓ^2-norm. We further show that several spatial difference schemes with jth-order ($j = 1, 2, 3, 4$) truncation error satisfy the assumptions required in our analysis. As a result, we obtain a series of temporally 2nd-order and spatially jth-order ($j = 1, 2, 3, 4$) unconditionally stable difference schemes for solving time-dependent Riesz space-fractional diffusion equations with variable coefficients. Numerical results are presented to illustrate our theoretical results.

Keywords Time-dependent space-fractional diffusion equation · Variable diffusion coefficients · High-order finite difference schemes · Stability · Convergence

Mathematics Subject Classification 26A33 · 35R11 · 65M06 · 65M12

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1 Introduction

We consider an initial-boundary value problem of Riesz space-fractional diffusion equation (RSFDE) [25]:

$$(\partial_t u)(\mathbf{x},t) = \sum_{i=1}^{l} d_i(\mathbf{x},t)(\partial_i^{\alpha_i} u)(\mathbf{x},t) + f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{1.1}$$

$$u(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times (0,T], \tag{1.2}$$

$$u(\mathbf{x},0) = \psi(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \tag{1.3}$$

where $\Omega = \prod_{i=1}^{l}(a_i,b_i) \subset \mathbb{R}^l$ is bounded, $\partial\Omega$ and $\bar{\Omega}$ denotes boundary, closure of Ω, respectively, $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$, $u(\mathbf{x},t)$ is unknown to be solved, $d_i(\mathbf{x},t)$ is positive over $\Omega \times (0,T]$ for $i = 1, 2, \ldots, l$, $f(\mathbf{x},t)$ and $\psi(\mathbf{x})$ are given source term and initial condition, respectively, $\partial_t u$ is the first-order partial derivative of u with respect to t, $\alpha_1, \alpha_2, \ldots, \alpha_l \in (1,2)$, $\partial_i^{\alpha_i} u$ is the Riesz fractional derivative of order α_i with respect to x_i defined by

$$\left(\partial_i^{\alpha_i} u\right)(\mathbf{x},t) := \sigma_{\alpha_i}\left({}_{a_i}D_{x_i}^{\alpha_i} u + {}_{x_i}D_{b_i}^{\alpha_i} u\right)(\mathbf{x},t), \quad \sigma_{\alpha_i} = -\frac{1}{2\cos(\frac{\pi\alpha_i}{2})} > 0, \tag{1.4}$$

$({}_{a_i}D_x^{\beta} u)(\mathbf{x},t)$ and $({}_{x}D_{b_i}^{\beta} u)(\mathbf{x},t)$ are left- and right- sided Riemann–Liouville (RL) derivatives defined respectively by

$$\left({}_{a_i}D_{x_i}^{\beta} u\right)(\mathbf{x},t) = \frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_{a_i}^{x_i} \frac{u(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_l, t)}{(x_i-\xi)^{\beta-n+1}} d\xi, \tag{1.5}$$

and

$$\left({}_{x_i}D_{b_i}^{\beta} u\right)(\mathbf{x},t) = \frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x_i^n}\int_{x_i}^{b_i} \frac{u(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_l, t)}{(\xi-x_i)^{\beta-n+1}} d\xi, \tag{1.6}$$

for some non-integer positive number $\beta > 0$ and $n = \lfloor\beta\rfloor + 1$. Here, $\Gamma(\cdot)$ denotes the gamma function. In particular, for a single variable function $w(x)$ on an interval $[x_L, x_R]$, its Riesz-derivative $\partial^\alpha w$ can be defined in a similar way.

In the last few decades, fractional calculus including fractional differentiation and integration has gained increasing attention and great importance due to its applications in various fields of science and engineering, such as biology, physics, control theory, electrical and mechanical engineering, data fitting; see [1,3,10,16,18,19]. As a class of fractional differential equations, RSFDEs have been widely and successfully used in modeling challenging phenomena such as long-range interactions, nonlocal dynamics [2,7,18]. To find efficient ways for solving RSFDEs naturally becomes an urgent topic. The closed-form analytical solutions of RSFDEs are usually unavailable. Fortunately, there has been some significant progress made by numerically solving the RSFDEs using finite difference schemes; see [4,6,8,9,11,13,15,20–25].

The advantage of solving the RSFDEs using high-order finite difference method is that the resulting discretized matrices from high-order schemes possess the same matrix structure and sparsity as the ones from low-order schemes, since matrices from both low- and high-order difference schemes always possess dense Toeplitz-like structure. That means high-order finite difference methods for solving RSFDEs do not require more computational cost and storage than low-order ones do.

Currently, there are two types of stability analysis for existing finite difference schemes for variable-coefficients RSFDEs. The first type stability is based on the spectral radius of the associated propagation matrix; see, e.g., [6,21]. Their stability analysis works only when $d_i(\mathbf{x}, t)$ $(1 \leq i \leq l)$ are proportional to each other. Moreover, the spectral-radius-based stability analysis may not induce a norm convergence proof. Therefore, it is natural to consider and study the second type stability that is based on the norm of the associated propagation matrix; see, e.g., [5,26]. In [26], the infinity-norm-based stability was studied and analyzed. Nevertheless, the infinity-norm-based analysis is valid only when a backward difference scheme is used for temporally discretization and the discretization matrix corresponding to $-\sum_{i=1}^{l} d_i(\mathbf{x}, t)\partial_i^{\alpha_i}$ is diagonally dominant with nonnegative diagonal entries. It is known that the diagonal-dominant property is valid for some first-order and second-order spatial discretization schemes; see [4,15,20,21]. However, such property no longer holds for higher-order spatial discretization schemes; see [6,21]. As a summary, the most accurate existing unconditionally stable difference schemes for (1.1)–(1.3) are only temporally 1st-order and spatially 2nd-order.

The main contribution of this paper is to study a framework for proving unconditional stability and convergence of temporally Crank–Nicolson scheme with high-order spatial discretization for the RSFDE (1.1)–(1.3). To the best of our knowledge, this is the first attempt to analyze stability and convergence of temporally 2nd-order and spatially high-order difference schemes for the variable-coefficients RSFDE (1.1)–(1.3). We show that if the spatial discretization matrix corresponding to the spatial operator $-\sum_{i=1}^{l} d_i(\mathbf{x}, t)\partial_i^{\alpha_i}$ is almost positive definite (i.e., the minimal eigenvalue of its Hermitian part is larger than a negative constant independent of mesh parameters), then the Crank–Nicolson scheme with such spatial discretization is unconditionally stable and convergent in the sense of discrete ℓ^2 norm. Furthermore, by exploiting Toeplitz-like structure of the spatial discretization matrix, we obtain two conditions which guarantee that the spatial discretization matrix is almost positive definite. The two conditions are, (a) $d_i(\mathbf{x}, t)$ is strictly positive, bounded and Lipschitz continuous with respect to the spatial variable x_i for $i = 1, 2, \ldots, l$, respectively; (b) the discretization matrix of $-\partial_i^{\alpha_i}$ is symmetric positive semi-definite and polynomial-decay with a decay order of $\alpha_i + 1$ for $i = 1, 2, \ldots, l$. A series of difference schemes with jth-order $(j = 1, 2, 3, 4)$ truncation error for the operators $-\partial_i^{\alpha_i}$ $(i = 1, 2, \ldots, l)$ is shown to satisfy the condition (b). As a result, we obtain a series of temporally 2nd-order and spatially jth-order $(j = 1, 2, 3, 4)$ unconditionally stable discretization schemes for RSFDEs (1.1)–(1.3). Numerical results are presented to support our theoretical analysis.

The outline of this paper is as follows. In Sect. 2, we present the Crank–Nicolson-type discretized RSFDE and establish its unconditional stability and convergence under the assumption that the spatial discretization matrix is almost positive definite. In Sect. 3, we exploit a Toeplitz-like structure of the spatial discretization matrix and prove the almost positive definiteness of the matrix. In Sect. 4, we verify a series of finite difference schemes with high-order truncation errors for $\partial_i^{\alpha_i}$, and establish stability and convergence of Crank–Nicolson-type discretized RSFDE with these high-order spatial discretization schemes. In Sect. 5, numerical experiments are reported to illustrate the theoretical results. Finally, we give some concluding remarks in Sect. 6.

2 Unconditional Stability and Convergence of Discretized RSFDEs

2.1 Crank–Nicolson Discretization

For positive integers $M_i (i = 1, 2, \ldots, l)$ and N, let $\tau = T/N$ and $h_i = (b_i - a_i)/(M_i + 1) (i = 1, 2, \ldots, l)$. Denote $t_0 = 0$, $t_n = n\tau$, $\bar{t}_n = (n - 0.5)\tau$ for $n = 1, \ldots, N$. Denote $x_{i,j} = a_i + jh_i$, for $0 \le j \le M_i + 1$, $1 \le i \le l$. Let $\mathbb{N}$ be the set of all integer numbers. For $i = 1, 2, \ldots, l$, define $\mathbb{I}_i = \{j \in \mathbb{N} | 1 \le j \le M_i\}$, $\hat{\mathbb{I}}_i = \{j \in \mathbb{N} | 0 \le j \le M_i + 1\}$, $\mathbb{K} = \mathbb{I}_1 \times \mathbb{I}_2 \times \cdots \times \mathbb{I}_l$, $\hat{\mathbb{K}} = \hat{\mathbb{I}}_1 \times \hat{\mathbb{I}}_2 \times \cdots \times \hat{\mathbb{I}}_l$, $\partial\mathbb{K} = \hat{\mathbb{K}} \backslash \mathbb{K}$. For a multiindex $J = (j_1, j_2, \ldots, j_l) \in \hat{\mathbb{K}}$, denote $\mathbf{x}_J = (x_{1,j_1}, x_{2,j_2}, \ldots, x_{l,j_l})$. Define grid function u_J^n, for $J \in \hat{\mathbb{K}}$, $0 \le n \le N$. For $J \in \mathbb{K}$ and $n = 1, 2, \ldots, N$, denote $d_{i,J}^n = d_i(\mathbf{x}_J, \bar{t}_n)$ and $f_J^n = f(\mathbf{x}_J, \bar{t}_n)$. Denote also $U_J^n = u(\mathbf{x}_J, t_n)$ for $J \in \hat{\mathbb{K}}$ and $n = 0, 1, \ldots, N$.

Let $\delta_i^{\alpha_i}$ be a linear discrete operator such that $\delta_i^{\alpha_i} U_J^n \approx (\partial_i^{\alpha_i} u)(\mathbf{x}_J, t_n)$ for all $J \in \mathbb{K}$, $n = 0, 1, \ldots, N$ and $i = 1, 2, \ldots, l$ (detailed definition of $\delta_i^{\alpha_i}$ will be given and studied in Sect. 3). For a grid function $\{y_J^n | J \in K, 0 \le n \le N\}$, we define discrete operators σ_t, δ_t as follows:

$$\sigma_t y_J^n = \frac{y_J^n + y_J^{n-1}}{2}, \quad \delta_t y_J^n = \frac{y_J^n - y_J^{n-1}}{\tau}, \quad n = 1, 2, \ldots, N, \quad J \in \mathbb{K}, \quad i = 1, 2, \ldots, l.$$

Then, a Crank–Nicolson-type discretized RSFDE for approximating (1.1)–(1.3) is given as follows

$$\delta_t u_J^n = \sum_{i=1}^{l} d_{i,J}^n \delta_i^{\alpha_i} \sigma_t u_J^n + f_J^n, \quad J \in \mathbb{K}, \quad 1 \le n \le N, \tag{2.1}$$

$$u_J^n = 0, \quad J \in \partial\mathbb{K}, \quad 1 \le n \le N, \tag{2.2}$$

$$u_J^0 = \psi(\mathbf{x}_J), \quad J \in \hat{\mathbb{K}}. \tag{2.3}$$

The solution u_J^n to the discretized RSFDE is an approximation to $u(\mathbf{x}_J, t_n)$ for all $J \in \mathbb{K}$, $1 \le n \le N$.

Let $\mathcal{G} = \{\mathbf{x}_J | J \in \mathbb{K}\}$ be the uniform spatial grid. Let $\mathbf{v}_G$ denote the vector obtained from arranging the entries of $\mathcal{G}$ in a certain ordering. For a function g defined on Ω, denote by $g(\mathbf{v}_G)$, the vector obtained from arranging values of g on $\mathbf{v}_G$ in the same ordering as that of $\mathbf{v}_G$. Then, the linear systems resulting from (2.1)–(2.3) are as follows (see, e.g., [4,6,21])

$$\left(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n\right)\mathbf{u}^n = \left(\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n\right)\mathbf{u}^{n-1} + \tau\mathbf{f}^n, \quad n = 1, 2, \ldots, N, \tag{2.4}$$

where $\mathbf{I}_k$ denotes $k \times k$ identity matrix, $\hat{M} = M_1 \times M_2 \times \cdots \times M_l$, $\mathbf{f}^n = f(\mathbf{v}_G, \bar{t}_n)$, $\mathbf{u}^n$ is unknown to be solved with its entries u_J^n $(J \in \mathbb{K})$ arranged in the same ordering as that of $\mathbf{v}_G$ for $n = 1, 2, \ldots, N$, $\mathbf{u}^0 = \psi(\mathbf{v}_G)$, $\mathbf{A}_n$ is the discretization matrix corresponding to the spatial operator $-\frac{1}{2}\sum_{i=1}^{l} d(\mathbf{x}, \bar{t}_n)\partial_i^{\alpha_i}$ such that $\mathbf{A}_n\mathbf{u}^n = \{-\frac{1}{2}\sum_{i=1}^{l} d_{i,J}^n \delta_i^{\alpha_i} u_J^n | J \in \mathbb{K}\}$ with its entries arranged in the same ordering as that of $\mathbf{v}_G$.

2.2 Unconditional Stability and Convergence

In this subsection, we study on the unique solvability, unconditional stability and convergence of the Crank–Nicolson-type RSFDE (2.4) by assuming that $\mathbf{A}_n$ is almost positive definite. For any symmetric matrices $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{R}^{m \times m}$, denote $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{H}_1$ if $\mathbf{H}_2 - \mathbf{H}_1$ is positive definite (or semi-definite). Especially, we denote $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{O}$, if $\mathbf{H}_2$ itself is positive definite (or semi-definite). Also, $\mathbf{H}_1 \prec$ (or $\preceq$) $\mathbf{H}_2$ and $\mathbf{O} \prec$ (or $\preceq$) $\mathbf{H}_2$ have the same meanings as those of $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{H}_1$ and $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{O}$, respectively.

For a matrix $\mathbf{C}$, denote by $\mathbf{C}^*$, the conjugate transpose of $\mathbf{C}$.

Lemma 1 *Assume $\mathbf{A}_n$ $(n = 1, 2, \ldots, N)$ are almost positive definite, i.e., there exists a positive constant c_0 independent of τ and h_i $(i = 1, 2, \ldots, l)$ such that $\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} \succeq -c_0\mathbf{I}_{\hat{M}}$ holds for $n = 1, 2, \ldots, N$. Then, whenever $\tau < c_0^{-1}$, the discretized RSFDE (2.4) is uniquely solvable and*

$$\max_{1\le n\le N} \sup_{\hat{M}>0} ||(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}||_2 \le (1 - c_0\tau)^{-\frac{1}{2}},$$

$$\max_{1\le n\le N} \sup_{\hat{M}>0} ||(\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)^{-1}(\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n)||_2 \le \left(\frac{1 + c_0\tau}{1 - c_0\tau}\right)^{\frac{1}{2}}.$$

Proof Denote $\mathbf{B}_n = \mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n$, $\mathbf{C}_n = \mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n$. By the assumption, for every $n \in \{1, 2, \ldots, N\}$ and every $\hat{M} > 0$, it holds

$$\mathbf{B}_n^{\mathrm{T}}\mathbf{B}_n = \mathbf{I}_{\hat{M}} + \tau(\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}}) + \tau^2\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n \succeq (1 - \tau c_0)\mathbf{I}_{\hat{M}}.$$

Thus, whenever $\tau < c_0^{-1}$, $\mathbf{B}_n$ is invertible and $||\mathbf{B}_n^{-1}||_2 = \sigma_{\min}(\mathbf{B}_n)^{-1} \le (1 - c_0\tau)^{-\frac{1}{2}}$, for every n, where $\sigma_{\min}(\cdot)$ denotes the minimal singular value of a matrix. On the other hand, from (2.4), we see that to prove the unique solvability is equivalent to prove the invertibility of $\mathbf{B}_n$ for every n. Therefore, $\tau < c_0^{-1}$ guarantees the unique solvability.

In the rest of the proof, we always assume $\tau < c_0^{-1}$ and $n \in \{1, 2, \ldots, N\}$ is a generic integer. For any $\hat{M} \times 1$ non-zero vector $\mathbf{z}$, denote

$$\mathbf{y} = \mathbf{B}_n^{-\mathrm{T}}\mathbf{z}, \quad q = \frac{\mathbf{y}^*\mathbf{y}}{\mathbf{y}^*\left(\mathbf{I}_{\hat{M}} + \tau^2\mathbf{A}_n\mathbf{A}_n^{\mathrm{T}}\right)\mathbf{y}}.$$

Then, by the assumption again,

$$\begin{aligned}
\frac{\mathbf{z}^*\mathbf{B}_n^{-1}\mathbf{C}_n\mathbf{C}_n^{\mathrm{T}}\mathbf{B}_n^{-\mathrm{T}}\mathbf{z}}{\mathbf{z}^*\mathbf{z}} = \frac{\mathbf{y}^*\mathbf{C}_n\mathbf{C}_n^{\mathrm{T}}\mathbf{y}}{\mathbf{y}^*\mathbf{B}_n\mathbf{B}_n^{\mathrm{T}}\mathbf{y}} &= \frac{\mathbf{y}^*\left[\mathbf{I}_{\hat{M}} - \tau\left(\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}}\right) + \tau^2\mathbf{A}_n\mathbf{A}_n^{\mathrm{T}}\right]\mathbf{y}}{\mathbf{y}^*\left[\mathbf{I}_{\hat{M}} + \tau\left(\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}}\right) + \tau^2\mathbf{A}_n\mathbf{A}_n^{\mathrm{T}}\right]\mathbf{y}} \\
&\le \frac{\mathbf{y}^*\left[\mathbf{I}_{\hat{M}} + \tau c_0\mathbf{I}_{\hat{M}} + \tau^2\mathbf{A}_n\mathbf{A}_n^{\mathrm{T}}\right]\mathbf{y}}{\mathbf{y}^*\left[\mathbf{I}_{\hat{M}} - \tau c_0\mathbf{I}_{\hat{M}} + \tau^2\mathbf{A}_n\mathbf{A}_n^{\mathrm{T}}\right]\mathbf{y}} \\
&= \frac{1 + c_0 q\tau}{1 - c_0 q\tau}.
\end{aligned}$$

Since $q \le 1$,

$$\frac{1 + c_0 q\tau}{1 - c_0 q\tau} = \frac{1 + c_0\tau}{1 - c_0\tau} + \frac{2c_0\tau(q - 1)}{(1 - c_0 q\tau)(1 - c_0\tau)} \le \frac{1 + c_0\tau}{1 - c_0\tau}.$$

Thus,

$$\frac{\mathbf{z}^*\mathbf{B}_n^{-1}\mathbf{C}_n\mathbf{C}_n^{\mathrm{T}}\mathbf{B}_n^{-\mathrm{T}}\mathbf{z}}{\mathbf{z}^*\mathbf{z}} \le \frac{1 + c_0\tau}{1 - c_0\tau},$$

which implies that $||\mathbf{B}_n^{-1}\mathbf{C}_n||_2 \le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{1}{2}}$. The proof is complete. □

Remark Notice that Lemma 1 requires almost positive definiteness of $\mathbf{A}_n$, i.e., $\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} \succeq -c_0\mathbf{I}_{\hat{M}}$ for mesh-parameters-independent constant c_0. Also, the analysis in the rest of this section depends heavily on this property. In Sects. 3 and 4, we will exploit a more detailed structure of $\mathbf{A}_n$ and show that $\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} \succeq -c_0\mathbf{I}_{\hat{M}}$ holds for almost all existing spatial discretization schemes. Thus, almost positive definiteness of $\mathbf{A}_n$ is an essential but very mild assumption in our analysis.

For an l-dimensional array, $\mathbf{y} = \{y_J | J \in \mathbb{K}\}$, define the discrete ℓ^2 norm $||\cdot||$ as

$$||\mathbf{y}|| = \left[\left(\prod_{i=1}^{l} h_i\right)\sum_{J\in\mathbb{K}} |y_J|^2\right]^{\frac{1}{2}}.$$

Lemma 2 *Assume that the assumption in Lemma 1 holds. Then, there exists a positive constant c_1 independent of τ, $h_i (i = 1, 2, \ldots, l)$ such that whenever $\tau \le c_1^{-1}$, the Crank–Nicolson-type discretized RSFDE (2.4) is unconditionally stable and*

$$\max_{1\le n\le N} ||\mathbf{u}^n|| \le 3^{\frac{c_1 T}{2}} \left(||\mathbf{u}^0|| + \tau \sum_{i=1}^{N} ||\mathbf{f}^i||\right),$$

where $c_1 = 2c_0$ with c_0 given by Lemma 1.

Proof Denote $\mathbf{B}_n = (\mathbf{I}_{\hat{M}} + \tau\mathbf{A}_n)$. Take $\tau \le c_1^{-1}$. Then, $\tau < c_0^{-1}$. By Lemma 1, $\mathbf{B}_n$ is invertible. Denote $\mathbf{E}_n = \mathbf{B}_n^{-1}(\mathbf{I}_{\hat{M}} - \tau\mathbf{A}_n)$. From (2.4), we see that $\mathbf{u}^n = \mathbf{E}_n\mathbf{u}^{n-1} + \tau\mathbf{B}_n^{-1}\mathbf{f}^n$, $n = 1, 2, \ldots, N$. Thus,

$$\begin{aligned}
\mathbf{u}^1 &= \mathbf{E}_1\mathbf{u}^0 + \tau\mathbf{B}_1^{-1}\mathbf{f}^1,\\
\mathbf{u}^2 &= \mathbf{E}_2\mathbf{u}^1 + \tau\mathbf{B}_2^{-1}\mathbf{f}^2 = \left(\prod_{i=1}^{2}\mathbf{E}_i\right)\mathbf{u}^0 + \tau\sum_{i=1}^{2}\left(\prod_{j=i+1}^{2}\mathbf{E}_j\right)\mathbf{B}_i^{-1}\mathbf{f}^i,\\
\mathbf{u}^3 &= \mathbf{E}_3\mathbf{u}^2 + \tau\mathbf{B}_3^{-1}\mathbf{f}^3 = \left(\prod_{i=1}^{3}\mathbf{E}_i\right)\mathbf{u}^0 + \tau\sum_{i=1}^{3}\left(\prod_{j=i+1}^{3}\mathbf{E}_j\right)\mathbf{B}_i^{-1}\mathbf{f}^i.
\end{aligned}$$

By induction, it is easy to check that

$$\mathbf{u}^n = \left(\prod_{i=1}^{n}\mathbf{E}_i\right)\mathbf{u}^0 + \tau\sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}\mathbf{E}_j\right)\mathbf{B}_i^{-1}\mathbf{f}^i, \quad n = 1, 2, \ldots, N,$$

where $\prod_{j=n+1}^{n}\mathbf{E}_j = \mathbf{I}_{\hat{M}}$ for $n = 1, 2, \ldots, N$. On the other hand, Lemma 1 gives $\max\limits_{1\le n\le N} ||\mathbf{E}_n||_2 \le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{1}{2}}$ and $\max\limits_{1\le n\le N} ||\mathbf{B}_n^{-1}||_2 \le \left(\frac{1}{1-c_0\tau}\right)^{\frac{1}{2}}$. Hence,

$$\begin{aligned}
||\mathbf{u}^n||_2 &\le \left(\prod_{i=1}^{n}||\mathbf{E}_i||_2\right)||\mathbf{u}^0||_2 + \tau\sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}||\mathbf{E}_j||_2\right)||\mathbf{B}_i^{-1}||_2||\mathbf{f}^i||_2\\
&\le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{n}{2}}||\mathbf{u}^0||_2 + \tau\sum_{i=1}^{n}\left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{n-i}{2}}\left(\frac{1}{1-c_0\tau}\right)^{\frac{1}{2}}||\mathbf{f}^i||_2\\
&\le \left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{N}{2}}\left(||\mathbf{u}^0||_2 + \tau\sum_{i=1}^{N}||\mathbf{f}^i||_2\right), \quad n = 1, 2, \ldots, N. \qquad (2.5)
\end{aligned}$$

Notice that $\left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{N}{2}} = \left[\left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{1}{c_0\tau}}\right]^{\frac{c_0 T}{2}}$. Denote $g(x) = \left(\frac{1+x}{1-x}\right)^{\frac{1}{x}}$. By checking derivatives of $\ln g(x)$, one can easily find that $g(x)$ is monotonically increasing over $x \in (0, \frac{1}{2}]$. Notice also the constraint $\tau \le c_1^{-1} = (2c_0)^{-1}$. Hence,

$$\left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{1}{c_0\tau}} \le \sup_{x\in(0,(2c_0)^{-1}]} \left(\frac{1+c_0x}{1-c_0x}\right)^{\frac{1}{c_0x}} = \sup_{x\in(0,\frac{1}{2}]} g(x) = g\left(\frac{1}{2}\right) = 9.$$

Thus, $\left(\frac{1+c_0\tau}{1-c_0\tau}\right)^{\frac{N}{2}} \le 3^{c_0T} = 3^{\frac{c_1T}{2}}$, which together with (2.5) implies that

$$\max_{1\le n\le N} ||\mathbf{u}^n||_2 \le 3^{\frac{c_1T}{2}} \left(||\mathbf{u}^0||_2 + \tau \sum_{i=1}^{N} ||\mathbf{f}^i||_2 \right).$$

Multiplying both sides of the above inequality by $(\prod_{i=1}^{l} h_i)^{\frac{1}{2}}$ leads to the desired result. □

For two numbers x, y, denote $x \lesssim$ (or $\gtrsim$) y if there exists a positive constant C independent of mesh parameters τ, h_i $(i = 1, 2, \ldots, l)$ such that $x \le$ (or $\ge$) Cy.

Denote by ∂_t^k, the kth-order partial derivative operator with respect to t. Especially, we simplify ∂_t^1 as ∂_t. Replacing u_J^n in (2.1)–(2.3) by U_J^n, we obtain through (1.1)–(1.3) that

$$\delta_t U_J^n = \sum_{i=1}^{l} d_{i,J}^n \delta_i^{\alpha_i} \sigma_t U_J^n + f_J^n + R_J^n, \quad J \in \mathbb{K}, \quad 1 \le n \le N, \tag{2.6}$$

$$U_J^n = 0, \quad J \in \partial\mathbb{K}, \quad 1 \le n \le N, \tag{2.7}$$

$$U_J^0 = \psi(\mathbf{x}_J), \quad J \in \hat{\mathbb{K}}, \tag{2.8}$$

with

$$R_J^n = \delta_t U_J^n - (\partial_t u)\left(\mathbf{x}_J, \bar{t}_n\right) + \sum_{i=1}^{l} d_{i,J}^n \left(\left(\partial_i^{\alpha_i} u\right)\left(\mathbf{x}_J, \bar{t}_n\right) - \delta_i^{\alpha_i} \sigma_t U_J^n \right).$$

Denote $e_J^n = U_J^n - u_J^n$ for $0 \le n \le N$, $J \in \hat{\mathbb{K}}$. Then, subtracting (2.1)–(2.3) from (2.6)–(2.8), we obtain error equations as follows

$$\delta_t e_J^n = \sum_{i=1}^{l} d_{i,J}^n \delta_i^{\alpha_i} \sigma_t e_J^n + R_J^n, \quad J \in \mathbb{K}, \quad 1 \le n \le N, \tag{2.9}$$

$$e_J^n = 0, \quad J \in \partial\mathbb{K}, \quad 1 \le n \le N, \tag{2.10}$$

$$e_J^0 = 0, \quad J \in \hat{\mathbb{K}}. \tag{2.11}$$

Denote the vectors, $\mathbf{e}^n = \{e_J^n | J \in \mathbb{K}\}$ and $\mathbf{R}^n = \{R_J^n | J \in \mathbb{K}\}$, with their entries arranged in the same ordering as that of $\mathbf{v}_G$. Define a set $C_t^k := \{g : \bar{\Omega} \times [0, T] \to \mathbb{R} | \partial_t^k g$ continuous on $\bar{\Omega} \times [0, T]\}$ for some positive integer k.

Lemma 3 *Assume the assumption in Lemma 1 holds and*

(a) $\max\limits_{1\le i\le l} \sup\limits_{(\mathbf{x},t)\in\Omega\times(0,T]} d_i(\mathbf{x}, t) \lesssim 1,$

(b) $u \in C_t^3$ *and* $\partial_i^{\alpha_i} u \in C_t^2$ *for* $i = 1, 2, \ldots, l,$

(c) *the spatial discretization* (3.1) *has* μ*th-order truncation error for some* $\mu > 0$, *i.e.*, $\max_{0\le n\le N}\max_{J\in\mathbb{K}}|(\partial_i^{\alpha_i}u)(\mathbf{x}_J,t_n)-\delta_i^{\alpha_i}U_J^n|\lesssim h_i^{\mu}$, $i=1,2,\ldots,l$.

Then, there exists a positive constant c *independent of* τ, h_i $(i=1,2,\ldots,l)$ *such that whenever* $\tau\le c$, *the discretized RSFDE* (2.4) *is convergent with*

$$\max_{1\le n\le N}||\mathbf{e}^n||\lesssim\tau^2+\sum_{i=1}^{l}h_i^{\mu}.$$

Proof Let $c=c_1^{-1}$ with c_1 given by Lemma 2. Let $\tau\le c$. Then, by Lemma 2 and $||\mathbf{e}^0||=0$,

$$\max_{1\le n\le N}||\mathbf{e}^n||\le\tau 3^{\frac{c_1T}{2}}\sum_{i=1}^{N}||\mathbf{R}^i||\lesssim\tau\sum_{i=1}^{N}||\mathbf{R}^i||. \tag{2.12}$$

Since $u\in C_t^3$, Taylor's expansion gives

$$U_J^n=u\left(\mathbf{x}_J,\bar{t}_n\right)+\sum_{i=1}^{2}\frac{\tau^i}{2^i i!}\left(\partial_t^i u\right)\left(\mathbf{x}_J,\bar{t}_n\right)+\int_{\bar{t}_n}^{t_n}\frac{(t_n-\xi)^2\left(\partial_t^3u\right)(\mathbf{x}_J,\xi)}{2!}d\xi,$$

$$U_J^{n-1}=u\left(\mathbf{x}_J,\bar{t}_n\right)+\sum_{i=1}^{2}\frac{(-\tau)^i}{2^i i!}\left(\partial_t^i u\right)\left(\mathbf{x}_J,\bar{t}_n\right)+\int_{\bar{t}_n}^{t_{n-1}}\frac{(t_{n-1}-\xi)^2\left(\partial_t^3u\right)(\mathbf{x}_J,\xi)}{2!}d\xi.$$

By $u\in C_t^3$ and above two equalities,

$$\begin{aligned}\left|\delta_tU_J^n-(\partial_tu)\left(\mathbf{x}_J,\bar{t}_n\right)\right|&=\frac{1}{\tau}\left|\int_{\bar{t}_n}^{t_n}\frac{(t_n-\xi)^2\left(\partial_t^3u\right)(\mathbf{x}_J,\xi)}{2!}d\xi\right.\\&\quad\left.-\int_{\bar{t}_n}^{t_{n-1}}\frac{(t_{n-1}-\xi)^2\left(\partial_t^3u\right)(\mathbf{x}_J,\xi)}{2!}d\xi\right|\\&\lesssim\frac{1}{\tau}\left[\int_{\bar{t}_n}^{t_n}(t_n-\xi)^2d\xi+\int_{t_{n-1}}^{\bar{t}_n}(t_{n-1}-\xi)^2d\xi\right]\lesssim\tau^2.\end{aligned} \tag{2.13}$$

Utilizing $\partial_i^{\alpha_i}u\in C_t^2$ and Taylor's expansion, we have

$$\left(\partial_i^{\alpha_i}u\right)\left(\mathbf{x}_J,\bar{t}_n\right)=\left(\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,t_n)+\int_{t_n}^{\bar{t}_n}\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi)d\xi,$$

$$\left(\partial_i^{\alpha_i}u\right)\left(\mathbf{x}_J,\bar{t}_n\right)=\left(\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,t_{n-1})+\int_{t_{n-1}}^{\bar{t}_n}\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi)d\xi.$$

Denote $\sigma_t(\partial_i^{\alpha_i}u)_J^n=\frac{1}{2}[(\partial_i^{\alpha_i}u)(\mathbf{x}_J,t_n)+(\partial_i^{\alpha_i}u)(\mathbf{x}_J,t_{n-1})]$. Then,

$$\begin{aligned}\left|\left(\partial_i^{\alpha_i}u\right)\left(\mathbf{x}_J,\bar{t}_n\right)-\sigma_t\left(\partial_i^{\alpha_i}u\right)_J^n\right|&=\frac{1}{2}\left|\int_{t_{n-1}}^{\bar{t}_n}\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi)d\xi+\int_{t_n}^{\bar{t}_n}\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi)d\xi\right|\\&=\frac{1}{2}\left|\int_{t_{n-1}}^{\bar{t}_n}\left[\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi)-\left(\partial_t\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\xi+\tau/2)\right]d\xi\right|\\&=\frac{1}{2}\left|\int_{t_{n-1}}^{\bar{t}_n}\int_{\xi+\frac{\tau}{2}}^{\xi}\left(\partial_t^2\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,\eta)d\eta d\xi\right|\lesssim\tau^2.\end{aligned}$$

By (**c**) and above inequality,

$$
\begin{aligned}
\left|\left(\partial_i^{\alpha_i}u\right)\left(\mathbf{x}_J,\bar{t}_n\right)-\delta_i^{\alpha_i}\sigma_t U_J^n\right| &= \left|\left(\partial_i^{\alpha_i}u\right)\left(\mathbf{x}_J,\bar{t}_n\right)-\sigma_t\left(\partial_i^{\alpha_i}u\right)_J^n+\sigma_t\left(\partial_i^{\alpha_i}u\right)_J^n-\delta_i^{\alpha_i}\sigma_t U_J^n\right| \\
&\lesssim \tau^2+\left|\sigma_t\left(\partial_i^{\alpha_i}u\right)_J^n-\delta_i^{\alpha_i}\sigma_t U_J^n\right| \\
&= \tau^2+\frac{1}{2}\left|\left(\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,t_n)-\delta_i^{\alpha_i}U_J^n\right. \\
&\quad \left.+\left(\partial_i^{\alpha_i}u\right)(\mathbf{x}_J,t_{n-1})-\delta_i^{\alpha_i}U_J^{n-1}\right| \\
&\lesssim \tau^2+h_i^{\mu}.
\end{aligned}
\tag{2.14}
$$

By (2.13), (2.14) and (**a**),

$$
\begin{aligned}
\max_{1\le n\le N}\max_{J\in\mathbb{K}}|R_J^n| \le \max_{1\le n\le N}\max_{J\in\mathbb{K}}&[|\delta_t U_J^n-(\partial_t u)(\mathbf{x}_J,\bar{t}_n)| \\
&+\sum_{i=1}^{l}d_{i,J}^n|(\partial_i^{\alpha_i}u)(\mathbf{x}_J,\bar{t}_n)-\delta_i^{\alpha_i}\sigma_t U_J^n|] \\
&\lesssim \tau^2+\sum_{i=1}^{l}d_{i,J}^n\left(\tau^2+h_i^{\mu}\right)\lesssim \tau^2+\sum_{i=1}^{l}h_i^{\mu}.
\end{aligned}
$$

By (2.12) and above inequality, we obtain

$$
\begin{aligned}
\max_{1\le n\le N}||\mathbf{e}^n|| \lesssim \tau\sum_{i=1}^{N}||\mathbf{R}^i|| &= \tau\sum_{i=1}^{N}\left[\left(\prod_{j=1}^{l}h_j\right)\sum_{J\in\mathbb{K}}|R_J^i|^2\right]^{\frac{1}{2}} \\
&\lesssim \tau\sum_{i=1}^{N}\left[\left(\prod_{j=1}^{l}h_j\right)\sum_{J\in\mathbb{K}}\left(\tau^2+\sum_{k=1}^{l}h_k^{\mu}\right)^2\right]^{\frac{1}{2}}\lesssim \tau^2+\sum_{k=1}^{l}h_k^{\mu},
\end{aligned}
$$

which completes the proof. □

3 Spatial Discretization Matrices

As we see from Sect. 2, an essential condition required in our analysis is that $\mathbf{A}_n$ is almost positive definite, i.e., $\mathbf{A}_n+\mathbf{A}_n^{\mathrm{T}}\succeq -c_0\mathbf{I}_{\hat{M}}$ for some mesh-parameters-independent positive constant c_0. In this section, we discuss the construction of Toeplitz-like structure of $\mathbf{A}_n$ and further exploit the Toeplitz-like structure to find some more specific and mild conditions that guarantee the almost positive definiteness of $\mathbf{A}_n$.

3.1 Toeplitz-Like Structure of $\mathbf{A}_n$

Let $v(x)$ be a real function vanishing at boundary of a bounded interval $[a,b]$. Let $h=(b-a)/(M+1)$ for some positive integer M. Without loss of generality, we assume the uniform-grid difference discretization of the Riesz derivative $\partial^{\alpha}v$ ($\alpha\in(1,2)$) to be of following form (see, e.g., [4,6,8,9,11,12,14,15,20–22,24])

$$
(\partial^{\alpha}v)(a+ih)\approx\delta^{\alpha}v(a+ih):=-\frac{1}{h^{\alpha}}\sum_{j=1}^{M}s_{|i-j|}^{(\alpha)}v(a+jh),\quad 1\le i\le M, \tag{3.1}
$$

where $s_k^{(\alpha_i)} (k \geq 0)$ are real numbers varying from different discretization schemes. Then, it is easy to see that (3.1) admits a Toeplitz matrix form such that

$$\mathbf{v}_1 = -\mathbf{S}\mathbf{v}_2 \tag{3.2}$$

with $\mathbf{v}_1 = (\delta^\alpha v(a+h), \delta^\alpha v(a+2h), \ldots, \delta^\alpha v(a+Mh))^{\mathrm{T}}$, $\mathbf{v}_2 = (v(a+h), v(a+2h), \ldots, v(a+Mh))^{\mathrm{T}}$,

$$\mathbf{S} = \frac{1}{h^\alpha} \begin{bmatrix} s_0^{(\alpha)} & s_1^{(\alpha)} & \cdots & s_{M-2}^{(\alpha)} & s_{M-1}^{(\alpha)} \\ s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} & \cdots & s_{M-2}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{M-2}^{(\alpha)} & \cdots & s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} \\ s_{M-1}^{(\alpha)} & s_{M-2}^{(\alpha)} & \cdots & s_1^{(\alpha)} & s_0^{(\alpha)} \end{bmatrix}.$$

For $J = (j_1, j_2, \ldots, j_l) \in \hat{\mathbb{K}}$, denote $J(i) = j_i$ and

$$\mathbb{S}_1(J) = \left(\bigcup_{j=1}^{M_1} \{j\} \right) \times \prod_{k=2}^{l} \{j_k\}, \quad \mathbb{S}_l(J) = \prod_{k=1}^{l-1} \{j_k\} \times \left(\bigcup_{j=1}^{M_l} \{j\} \right),$$

$$\mathbb{S}_i(J) = \prod_{k=1}^{i-1} \{j_k\} \times \left(\bigcup_{j=1}^{M_i} \{j\} \right) \times \prod_{k=i+1}^{l} \{j_k\}, \quad i = 2, \ldots, l-1.$$

Then, for an l-dimensional grid function $\{y_J | J \in \mathbb{K}\}$, (1.4)–(1.6) and (3.1) induce a definition of $\delta_i^{\alpha_i}$ with following form

$$\delta_i^{\alpha_i} y_J = -\frac{1}{h_i^{\alpha_i}} \sum_{I \in \mathbb{S}_i(J)} s_{|I(i)-J(i)|}^{(\alpha_i)} y_I, \quad J \in \mathbb{K}, \quad i = 1, 2, \ldots, l. \tag{3.3}$$

For a q-dimensional array $\mathcal{Y} = \{y_{i_1, i_2, \ldots, i_q} | 1 \leq i_j \leq m_j, \ 1 \leq j \leq q\}$, define its lexicographic ordering as

$$y_{i_1, i_2, \ldots, i_q} \prec y_{j_1, j_2, \ldots, j_q} \Leftrightarrow \begin{cases} i_1 < j_1, & \text{or} \\ \exists m \geq 2 & s.t. \ i_k = j_k \text{ for } k = 1, 2, \ldots m-1 \text{ and } i_m < j_m. \end{cases}$$

Also, for above q-dimensional array $\mathcal{Y}$, denote by $\mathcal{V}(\mathcal{Y})$, the vector obtained from arranging entries of $\mathcal{Y}$ in the lexicographic ordering.

In the rest of this paper, we assume $\mathbf{v}_G = \mathcal{V}(\mathcal{G})$ and definitions of the discrete operators $\delta_i^{\alpha_i}$ $(i = 1, 2, \ldots, l)$ in (2.1) are given by (3.3). Denote $M_i^- = \prod_{j=1}^{i-1} M_j$, $M_i^+ = \prod_{j=i+1}^{l} M_j$, for $i = 1, 2, \ldots, l$. Especially, $M_1^- = M_l^+ = 1$. Then, from (3.2), we see that the matrices $\mathbf{A}_n$ $(n = 1, 2, \ldots, N)$ have following Toeplitz-like structure

$$\mathbf{A}_n = \frac{1}{2} \sum_{i=1}^{l} \mathbf{D}_{i,n} \mathbf{T}_i, \quad \mathbf{T}_i = \mathbf{I}_{M_i^-} \otimes \mathbf{S}_i \otimes \mathbf{I}_{M_i^+}, \quad n = 1, 2, \ldots, N, \tag{3.4}$$

where $\mathbf{D}_{i,n} = d_i(\mathbf{v}_G, \bar{t}_n) = \mathcal{V}(\{d^n_{i,J} | J \in \mathbb{K}\})$, "$\otimes$" denotes the Kronecker product,

$$\mathbf{S}_i = \frac{1}{h_i^{\alpha_i}} \begin{bmatrix} s_0^{(\alpha_i)} & s_1^{(\alpha_i)} & \cdots & s_{M_i-2}^{(\alpha_i)} & s_{M_i-1}^{(\alpha_i)} \\ s_1^{(\alpha_i)} & s_0^{(\alpha_i)} & s_1^{(\alpha_i)} & \cdots & s_{M_i-2}^{(\alpha_i)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{M_i-2}^{(\alpha_i)} & \cdots & s_1^{(\alpha_i)} & s_0^{(\alpha_i)} & s_1^{(\alpha_i)} \\ s_{M_i-1}^{(\alpha_i)} & s_{M_i-2}^{(\alpha_i)} & \cdots & s_1^{(\alpha_i)} & s_0^{(\alpha_i)} \end{bmatrix}.$$

For the term 'Toeplitz-like', readers may refer to [17]. In the coming subsection, we utilize the Toeplitz-like structure of $\mathbf{A}_n$ to find some more specific conditions guaranteeing $\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} \succeq -c_0 \mathbf{I}_{\hat{M}}$.

3.2 Almost Positive Definiteness of $\mathbf{A}_n$

Define a set of sequences as

$$\mathcal{D}_s := \left\{ \{w_k\}_{k\geq 0} \;\middle|\; ||\{w_k\}||_{\mathcal{D}_s} := \sup_{k\geq 0} |w_k|(1+k)^{1+s} < +\infty \right\},$$

for some $s > 0$. Then, it is easy to check that $\mathcal{D}_s$ is a linear normed space equipped with norm $||\cdot||_{\mathcal{D}_s}$. Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ real matrices. For a diagonal matrix $\mathbf{Z} = diag(z_1, z_2, \ldots, z_m) \in \mathbb{R}^{m\times m}$, denote

$$\min(\mathbf{Z}) = \min_{1\leq i\leq m} z_i, \quad \nabla(\mathbf{Z}) := \max_{1\leq i,j\leq m,\ i\neq j} \frac{|z_i - z_j|}{|i-j|}.$$

For a symmetric matrix $\mathbf{H} \in \mathbb{R}^{m\times m}$ and a nonnegative diagonal matrix $\mathbf{Z} \in \mathbb{R}^{m\times m}$, denote

$$\Delta_{\mathbf{H}}(\mathbf{Z}) := \mathbf{ZH} + \mathbf{HZ} - 2\mathbf{Z}^{\frac{1}{2}}\mathbf{H}\mathbf{Z}^{\frac{1}{2}}.$$

Lemma 4 *Let* $\mathbf{Z} = diag(z_1, z_2, \ldots, z_M) \in \mathbb{R}^{M\times M}$. *Assume*

(i) there exists a positive number $\check{z}$ *such that* $\min(\mathbf{Z}) \geq \check{z}$*,*
(ii) there exists a positive number $\tilde{z}$ *such that* $\nabla(\mathbf{Z}) \leq \tilde{z}h$*,*
(iii) there exists a positive number c_0 *such that* $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \leq c_0$.

Then, $||\Delta_{\mathbf{S}}(\mathbf{Z})||_2 \leq \frac{c_0 \tilde{z}^2 (b-a)^{2-\alpha}}{2\check{z}(2-\alpha)}$.

Proof It is easy to check the (i, j)-th entry r_{ij} of $\Delta_{\mathbf{S}}(\mathbf{Z})$ is given by

$$r_{ij} = h^{-\alpha}\left(z_i s_{|i-j|}^{(\alpha)} + s_{|i-j|}^{(\alpha)} z_j - 2 z_i^{\frac{1}{2}} s_{|i-j|}^{(\alpha)} z_j^{\frac{1}{2}}\right) = h^{-\alpha}\left(z_i^{\frac{1}{2}} - z_j^{\frac{1}{2}}\right)^2 s_{|i-j|}^{(\alpha)}, \quad 1 \leq i, j \leq M.$$

Using assumptions (**i**)–(**iii**), it holds

$$\begin{aligned}
|r_{ij}| &= \frac{1}{h^{\alpha}}\left|s^{(\alpha)}_{|i-j|}\right|\left|z_i^{\frac{1}{2}}-z_j^{\frac{1}{2}}\right|^2\\
&= \frac{1}{h^{\alpha}}\left|s^{(\alpha)}_{|i-j|}\right|\left|\int_{z_i}^{z_j}\frac{1}{2}\xi^{-\frac{1}{2}}d\xi\right|^2\\
&\le \frac{1}{h^{\alpha}}\left|s^{(\alpha)}_{|i-j|}\right|\left|\int_{z_i}^{z_j}\frac{1}{2}\check{z}^{-\frac{1}{2}}d\xi\right|^2\\
&= 4^{-1}\check{z}^{-1}h^{-\alpha}\left|s^{(\alpha)}_{|i-j|}\right|\left|z_i-z_j\right|^2\\
&\le 4^{-1}\check{z}^{-1}h^{2-\alpha}\left|s^{(\alpha)}_{|i-j|}\right|\tilde{z}^2|i-j|^2\\
&\le \frac{c_0\tilde{z}^2h^{2-\alpha}|i-j|^2}{4\check{z}(1+|i-j|)^{\alpha+1}}.
\end{aligned}\tag{3.5}$$

Using (3.5), we obtain

$$\begin{aligned}
||\Delta_{\mathbf{S}}(\mathbf{Z})||_{\infty} = \max_{1\le i\le M}\sum_{j=1}^{M}|r_{ij}| &= \max_{1\le i\le M}\left(|r_{ii}|+\sum_{j=1}^{i-1}|r_{ij}|+\sum_{j=i+1}^{M}|r_{ij}|\right)\\
&\le \max_{1\le i\le M}\frac{c_0\tilde{z}^2h^{2-\alpha}}{4\check{z}}\left(\sum_{j=1}^{i-1}\frac{|i-j|^2}{(1+|i-j|)^{\alpha+1}}\right.\\
&\quad\left.+\sum_{j=i+1}^{M}\frac{|i-j|^2}{(1+|i-j|)^{\alpha+1}}\right)\\
&= \max_{1\le i\le M}\frac{\tilde{z}^2c_0h^{2-\alpha}}{4\check{z}}\left(\sum_{k=1}^{i-1}\frac{k^2}{(1+k)^{\alpha+1}}+\sum_{k=1}^{M-i}\frac{k^2}{(1+k)^{\alpha+1}}\right)\\
&\le \frac{\tilde{z}^2c_0h^{2-\alpha}}{2\check{z}}\sum_{k=1}^{M}k^{1-\alpha}\\
&\le \frac{\tilde{z}^2c_0h^{2-\alpha}}{2\check{z}}\sum_{k=1}^{M}\int_{k-1}^{k}x^{1-\alpha}dx\\
&= \frac{c_0\tilde{z}^2M^{2-\alpha}}{2\check{z}(2-\alpha)}\left(\frac{b-a}{M+1}\right)^{2-\alpha}\le\frac{c_0\tilde{z}^2(b-a)^{2-\alpha}}{2\check{z}(2-\alpha)}.
\end{aligned}$$

Since $\mathbf{S}$ is symmetric, $\Delta_{\mathbf{S}}(\mathbf{Z})$ is also symmetric. Therefore, we have $||\Delta_{\mathbf{S}}(\mathbf{Z})||_2 = \rho(\Delta_{\mathbf{S}}(\mathbf{Z})) \le ||\Delta_{\mathbf{S}}(\mathbf{Z})||_{\infty}$, which together with the above inequality completes the proof. □

To extend Lemma 4 to multidimensional case, we firstly introduce several notations. Let $\{d_J|J\in\mathbb{K}\}$ be a randomly given real l-dimensional array. For $\mathbf{D}=diag(\mathcal{V}(\{d_J|J\in\mathbb{K}\}))$, denote

$$\nabla_i(\mathbf{D}) := \max_{J\in\mathbb{K}}\ \max_{I,K\in\mathbb{S}_i(J),\ I\ne K}\frac{|d_I-d_K|}{|I(i)-K(i)|},\quad i=1,2,\ldots,l.$$

For $J = (i_1, i_2, \ldots, i_l) \in \mathbb{K}$ and $j = 1, 2, \ldots, l$, define mappings $E_{j\leftrightarrow l}(\cdot)$ such that

$$E_{j\leftrightarrow l}(J) = (i_1, i_2, \ldots, i_{j-1}, i_l, i_{j+1}, \ldots, i_{l-1}, i_j).$$

For $k = 1, 2, \ldots, l$, define sets of multiindices such that

$$\mathbb{K}_{k\leftrightarrow l} := \{(i_1, i_2, \ldots, i_l)|\text{ for } m \neq k \text{ and } m \neq l, \quad 1 \leq i_m \leq M_m, \quad 1 \leq i_k \leq M_l, \quad 1 \leq i_l \leq M_k\}.$$

Letting $\mathbf{v} = \mathcal{V}(\{v_J | J \in \mathbb{K}\})$ and $\mathbf{y} = \mathcal{V}(\{y_J | J \in \mathbb{K}_{k\leftrightarrow l}\})$ with $1 \leq k \leq l$ be two arbitrary l-dimensional array, define permutation matrix $\mathbf{P}_k$ such that

$$\mathbf{y} = \mathbf{P}_k \mathbf{v} \Leftrightarrow y_{E_{k\leftrightarrow l}(J)} = v_J, \quad \forall J \in \mathbb{K}. \tag{3.6}$$

Lemma 5 *Let* $\mathbf{D} = diag(\mathcal{V}(\{d_J | J \in \mathbb{K}\})) \in \mathbb{R}^{\hat{M}\times\hat{M}}$. *For any* $i \in \{1, 2, \ldots, l\}$, *assume*

(i) $\min(\mathbf{D}) \gtrsim 1$,
(ii) $\nabla_i(\mathbf{D}) \lesssim h_i$,
(iii) $\{s_k^{(\alpha_i)}\}_{k\geq 0} \in \mathcal{D}_{\alpha_i}$.

Then, $||\Delta_{\mathbf{T}_i}(\mathbf{D})||_2 \lesssim 1$.

Proof Denote $\tilde{\mathbf{D}} = \mathbf{P}_i \mathbf{D} \mathbf{P}_i^{\mathrm{T}}$, $\tilde{\mathbf{T}}_i = \mathbf{P}_i \mathbf{T}_i \mathbf{P}_i^{\mathrm{T}}$, $\mathring{M}_i = M_i^- M_i^+$. Then, it is easy to see that $\tilde{\mathbf{T}}_i = \mathbf{I}_{\mathring{M}_i} \otimes \mathbf{S}_i$. Rewrite $\tilde{\mathbf{D}}$ as $\tilde{\mathbf{D}} = diag(\tilde{\mathbf{D}}_1, \tilde{\mathbf{D}}_2, \ldots, \tilde{\mathbf{D}}_{\mathring{M}_i})$ with diagonal matrices $\tilde{\mathbf{D}}_j \in \mathbb{R}^{M_i \times Mi}$ for $j = 1, 2, \ldots, \mathring{M}_i$. Then, it is easy to check that $\max_{1\leq j\leq \mathring{M}_i} \nabla(\tilde{\mathbf{D}}_j) = \nabla_i(\mathbf{D})$. Thus, by **(ii)**,

$$\max_{1\leq j\leq \mathring{M}_i} \nabla(\tilde{\mathbf{D}}_j) \lesssim h_i. \tag{3.7}$$

Moreover, it follows from **(i)** and $\min(\tilde{\mathbf{D}}) = \min(\mathbf{D})$ that

$$\min_{1\leq j\leq \mathring{M}_i} \min(\tilde{\mathbf{D}}_j) = \min(\mathbf{D}) \gtrsim 1. \tag{3.8}$$

Applying Lemma 4 to (3.7), (3.8) and **(iii)**, we obtain

$$\max_{1\leq j\leq \mathring{M}_i} \left|\left|\Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_j)\right|\right|_2 \lesssim 1.$$

Since $\mathbf{D}$ is positive diagonal matrix, so is $\tilde{\mathbf{D}}$. Then,

$$\begin{aligned}
\Delta_{\mathbf{T}_i}(\mathbf{D}) &= \mathbf{P}_i^{\mathrm{T}} \left(\mathbf{P}_i \Delta_{\mathbf{T}_i}(\mathbf{D}) \mathbf{P}_i^{\mathrm{T}}\right) \mathbf{P}_i \\
&= \mathbf{P}_i^{\mathrm{T}} \left(\mathbf{P}_i \mathbf{D} \mathbf{P}_i^{\mathrm{T}} \mathbf{P}_i \mathbf{T}_i \mathbf{P}_i^{\mathrm{T}} + \mathbf{P}_i \mathbf{T}_i \mathbf{P}_i^{\mathrm{T}} \mathbf{P}_i \mathbf{D} \mathbf{P}_i^{\mathrm{T}} - 2\mathbf{P}_i \mathbf{D}^{\frac{1}{2}} \mathbf{P}_i^{\mathrm{T}} \mathbf{P}_i \mathbf{T}_i \mathbf{P}_i^{\mathrm{T}} \mathbf{P}_i \mathbf{D}^{\frac{1}{2}} \mathbf{P}_i^{\mathrm{T}}\right) \mathbf{P}_i \\
&= \mathbf{P}_i^{\mathrm{T}} \left(\tilde{\mathbf{D}} \tilde{\mathbf{T}}_i + \tilde{\mathbf{T}}_i \tilde{\mathbf{D}} - 2\tilde{\mathbf{D}}^{\frac{1}{2}} \tilde{\mathbf{T}}_i \tilde{\mathbf{D}}^{\frac{1}{2}}\right) \mathbf{P}_i \\
&= \mathbf{P}_i^{\mathrm{T}} diag\left(\Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_1), \Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_2), \ldots, \Delta_{\mathbf{S}_i}\left(\tilde{\mathbf{D}}_{\mathring{M}_i}\right)\right) \mathbf{P}_i.
\end{aligned}$$

Thus,

$$\begin{aligned}
||\Delta_{\mathbf{T}_i}(\mathbf{D})||_2 &\leq \left|\left|\mathbf{P}_i^{\mathrm{T}}\right|\right|_2 \left|\left|diag\left(\Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_1), \Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_2), \ldots, \Delta_{\mathbf{S}_i}\left(\tilde{\mathbf{D}}_{\mathring{M}_i}\right)\right)\right|\right|_2 ||\mathbf{P}_i||_2 \\
&= \max_{1\leq j\leq \mathring{M}_i} \left|\left|\Delta_{\mathbf{S}_i}(\tilde{\mathbf{D}}_j)\right|\right|_2 \lesssim 1,
\end{aligned}$$

which completes the proof. □

For $\mathbf{x} = (x_1, x_2, \ldots, x_l),\ \mathbf{y} = (y_1, y_2, \ldots, y_l) \in \Omega$, denote

$$|\mathbf{x} - \mathbf{y}| = \left(\sum_{i=1}^{l} |x_i - y_i|^2\right)^{\frac{1}{2}}, \quad \mathcal{S}_1(\mathbf{x}) = (a_1, b_1) \times \prod_{k=2}^{l} \{x_k\}, \quad \mathcal{S}_l(\mathbf{x}) = \prod_{k=1}^{l-1} \{x_k\} \times (a_l, b_l),$$

$$\mathcal{S}_i(\mathbf{x}) = \prod_{k=1}^{i-1} \{x_k\} \times (a_i, b_i) \times \prod_{k=i+1}^{l} \{x_k\}, \quad i = 2, 3, \ldots, l-1.$$

Define the set of functions that are Lipschitz continuous with respect to the spatial variable x_i $(i = 1, 2, \ldots, l)$ as

$$\mathcal{L}_i(\Omega) := \left\{ w(\mathbf{x}) \,\middle|\, |w|_{\mathcal{L}_i(\Omega)} := \sup_{\mathbf{x} \in \Omega} \sup_{\mathbf{y}, \mathbf{z} \in \mathcal{S}_i(\mathbf{x}),\ \mathbf{y} \neq \mathbf{z}} \frac{|w(\mathbf{y}) - w(\mathbf{z})|}{|\mathbf{y} - \mathbf{z}|} < +\infty \right\}.$$

With Lemma 5, we obtain following lemma that guarantees $\mathbf{A}_n$ $(n = 1, 2, \ldots, N)$ are almost positive definite.

Lemma 6 *Assume*

(d) for $i = 1, 2, \ldots, l$, $\{s_k^{(\alpha_i)}\}_{k \geq 0} \in \mathcal{D}_{\alpha_i}$,
(e) for $i = 1, 2, \ldots, l$, $\mathbf{S}_i \succeq \mathbf{O}$,
(f) $\min\limits_{1 \leq i \leq l} \inf\limits_{(\mathbf{x}, t) \in \Omega \times (0, T]} d_i(\mathbf{x}, t) \gtrsim 1$,
(g) for any $t \in (0, T]$ *and* $i = 1, 2, \ldots, l$, $d_i(\cdot, t) \in \mathcal{L}_i(\Omega)$ *with* $\sup\limits_{t \in (0, T]} \max\limits_{1 \leq i \leq l} |d_i(\cdot, t)|_{\mathcal{L}_i(\Omega)} \lesssim 1$.

Then, $\mathbf{A}_n$ $(n = 1, 2, \ldots, N)$ *are almost positive definite, i.e., there exists a positive constant* c_0 *independent of* τ, $h_i (i = 1, 2, \ldots, l)$ *such that* $\mathbf{A}_n + \mathbf{A}_n^{\mathrm{T}} \succeq -c_0 \mathbf{I}_{\hat{M}}$ *for* $n = 1, 2, \ldots, N$.

Proof By (**f**),

$$\min_{1 \leq i \leq l} \min_{1 \leq n \leq N} \min(\mathbf{D}_{i,n}) \gtrsim 1. \tag{3.9}$$

For $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$, denote $\mathbf{x}(i) = x_i$. Recall that $\mathcal{G} = \{\mathbf{x}_J | J \in \mathbb{K}\} \subset \Omega$. Moreover, from definitions of $\mathcal{S}_i(\cdot)$ and $\mathbb{S}_i(\cdot)$, we see that for any $J \in \mathbb{K}$,

$$\{\mathbf{x}_I | I \in \mathbb{S}_i(J)\} = \mathcal{S}_i(\mathbf{x}_J) \cap \mathcal{G}, \quad i = 1, 2, \ldots, l.$$

Thus, from (**g**), we see that for every $n \in \{1, 2, \ldots, N\}$ and every $i \in \{1, 2, \ldots, l\}$, it holds

$$\begin{aligned}
1 \gtrsim |d_i(\cdot, \bar{t}_n)|_{\mathcal{L}_i(\Omega)} &= \sup_{\mathbf{x} \in \Omega} \sup_{\mathbf{y}, \mathbf{z} \in \mathcal{S}_i(\mathbf{x}),\ \mathbf{y} \neq \mathbf{z}} \frac{|d_i(\mathbf{y}, \bar{t}_n) - d_i(\mathbf{z}, \bar{t}_n)|}{|\mathbf{y} - \mathbf{z}|} \\
&\geq \max_{\mathbf{x} \in \mathcal{G}} \sup_{\mathbf{y}, \mathbf{z} \in \mathcal{S}_i(\mathbf{x}),\ \mathbf{y} \neq \mathbf{z}} \frac{|d_i(\mathbf{y}, \bar{t}_n) - d_i(\mathbf{z}, \bar{t}_n)|}{|\mathbf{y} - \mathbf{z}|} \\
&= \max_{J \in \mathbb{K}} \sup_{\mathbf{y}, \mathbf{z} \in \mathcal{S}_i(\mathbf{x}_J),\ \mathbf{y} \neq \mathbf{z}} \frac{|d_i(\mathbf{y}, \bar{t}_n) - d_i(\mathbf{z}, \bar{t}_n)|}{|\mathbf{y} - \mathbf{z}|} \\
&\geq \max_{J \in \mathbb{K}} \max_{I, K \in \mathbb{S}_i(J),\ I \neq K} \frac{|d_i(\mathbf{x}_I, \bar{t}_n) - d_i(\mathbf{x}_K, \bar{t}_n)|}{|\mathbf{x}_I - \mathbf{x}_K|} \\
&= \max_{J \in \mathbb{K}} \max_{I, K \in \mathbb{S}_i(J),\ I \neq K} \frac{|d_i(\mathbf{x}_I, \bar{t}_n) - d_i(\mathbf{x}_K, \bar{t}_n)|}{|I(i) - K(i)| h_i} = h_i^{-1} \nabla_i(\mathbf{D}_{i,n}).
\end{aligned}$$

Therefore,

$$\max_{1\le n\le N} \nabla_i(\mathbf{D}_{i,n}) \lesssim h_i, \quad i=1,2,\ldots,l. \tag{3.10}$$

Applying Lemma 5 to (3.9), (3.10) and (**d**), we obtain $\max\limits_{1\le n\le N}\max\limits_{1\le i\le l} ||\Delta_{\mathbf{T}_i}(\mathbf{D}_{i,n})||_2 \lesssim 1$. On the other hand, it is noticeable that $\mathbf{A}_n+\mathbf{A}_n^{\mathrm{T}} = \sum_{i=1}^{l}\mathbf{D}_{i,n}^{\frac{1}{2}}\mathbf{T}_i\mathbf{D}_{i,n}^{\frac{1}{2}} + \frac{1}{2}\sum_{i=1}^{l}\Delta_{\mathbf{T}_i}(\mathbf{D}_{i,n})$. From (**e**), it is easy to see that $\mathbf{T}_i \succeq \mathbf{O}$ and thus $\sum_{i=1}^{l}\mathbf{D}_{i,n}^{\frac{1}{2}}\mathbf{T}_i\mathbf{D}_{i,n}^{\frac{1}{2}} \succeq \mathbf{O}$. Take $c_0 = 1+\frac{1}{2}\sum_{i=1}^{l}\max\limits_{1\le n\le N}||\Delta_{\mathbf{T}_i}(\mathbf{D}_{i,n})||_2$. Then, we obtain $\mathbf{A}_n+\mathbf{A}_n^{\mathrm{T}} \succeq \sum_{i=1}^{l}\mathbf{D}_{i,n}^{\frac{1}{2}}\mathbf{T}_i\mathbf{D}_{i,n}^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{l}||\Delta_{\mathbf{T}_i}(\mathbf{D}_{i,n})||_2\mathbf{I}_{\hat{M}} \succeq -c_0\mathbf{I}_{\hat{M}}$, which completes the proof. □

Remark Lemma 6 presents four assumptions (**d**)–(**g**) to guarantee the almost positive definiteness of $\mathbf{A}_n$. Among these assumptions, (**f**) and (**g**) are mild and specific, which only requires $d_i(\mathbf{x},t)$ to be bounded above and below by positive constants, and Lipschitz-continuous with respect to x_i, for $i=1,2,\ldots,l$, respectively. Notice also that the assumptions (**d**) and (**e**) are unspecified, both of which are related to the spatial discretization (3.1) for the Riesz derivative ∂^α. In the next section, we will verify a series of finite difference schemes with form (3.1) to show that these schemes satisfy (**d**) and (**e**).

4 Numerical Schemes for Riesz Derivatives

In this section, we show that a series of spatial difference schemes satisfy (**d**) and (**e**). To verify (**d**) and (**e**), it suffices to show

$$\mathbf{S} \succeq \mathbf{O}, \qquad \forall \alpha \in (1,2), \tag{4.1}$$

$$\{s_k^{(\alpha)}\}_{k\ge 0} \in \mathcal{D}_\alpha, \quad \forall \alpha \in (1,2), \tag{4.2}$$

Let us first introduce some notations. Let $\{y_k\}_{k\ge0}$ and $\{z_k\}_{k\ge0}$ denote two sequences. For some nonnegative integer m, define operators $S_{\pm m}(\cdot)$, $F_{\pm m}(\cdot)$, respectively as

$$\begin{aligned}
\{z_k\}_{k\ge0} = S_m(\{y_k\}_{k\ge0}) &\iff z_k = y_{k+m}, \quad k\ge 0,\\
\{z_k\}_{k\ge0} = S_{-m}(\{y_k\}_{k\ge0}) &\iff z_k = 0,\ 0\le k\le m-1 \text{ and } z_k = y_{k-m}, \quad k\ge m,\\
\{z_k\}_{k\ge0} = F_m(\{y_k\}_{k\ge0}) &\iff z_k = y_{m-k},\ 0\le k\le m \text{ and } z_k = 0, \quad k>m,\\
\{z_k\}_{k\ge0} = F_{-m}(\{y_k\}_{k\ge0}) &\iff z_k = 0, \quad k\ge 0, \quad m\ge 1.
\end{aligned}$$

For any sequences $\{y_k\}_{k\ge0}$ and for some integer m, define the operator

$$\mathcal{R}_m^\alpha(\{y_k\}_{k\ge0}) = \frac{-[S_m(\{y_k\}_{k\ge0}) + F_m(\{y_k\}_{k\ge0})]}{2\cos(\pi\alpha/2)}, \quad \alpha\in(1,2).$$

To conclude the theoretical results in Sects. 2 and 3, we have following theorem.

Theorem 7 *Assume the assumptions* (**d**)–(**g**) *hold. Let the mesh-parameters-independent positive constant* $c_1 = 2c_0$ *with* c_0 *given by Lemma* 6. *Take* $\tau \le c_1^{-1}$. *Then,*

(i) (2.4) *with spatial discretization of form* (3.4) *is uniquely solvable and unconditionally stable in the sense that*

$$\max_{1\le n\le N} ||\mathbf{u}^n|| \le 3^{\frac{c_1 T}{2}}\left(||\mathbf{u}^0|| + \tau\sum_{i=1}^{N}||\mathbf{f}^i||\right),$$

(ii) if in addition (**a**)–(**c**) *hold, then* (2.4) *with* (3.4) *is convergent with*

$$\max_{1\le n\le N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i^{\mu}.$$

Proof The results follow from applying Lemma 6 to Lemmas 1–3. □

For convenience of statement, we use the notation (**q**) to represent the four specified assumptions (**a**), (**b**), (**f**) and (**g**) in the rest of this paper.

4.1 Verification of Schemes from [15,21]

In this subsection, we verify the conditions (4.1)–(4.2) for the shifted Grünwald formula from [15] and two weighted-shifted Grünwald formulas from [21]. Let

$$g_0^{(\alpha)} = -1, \quad g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha+1}{k+1}\right) g_k^{(\alpha)}, \quad \alpha \in (1,2), \quad k = 0, 1, 2, \ldots \tag{4.3}$$

Then, $\{s_k^{(\alpha)}\}_{k\ge 0}$ resulting from [15] is given by

$$\{s_k^{(\alpha)}\}_{k\ge 0} = \mathcal{R}_1(\{g_k^{(\alpha)}\}_{k\ge 0}). \tag{4.4}$$

Lemma 8 [21,22] *For any* $\alpha \in (1,2)$*, it holds*

(i) $g_1^{(\alpha)} > 0, \;\; g_0^{(\alpha)} < g_2^{(\alpha)} < g_3^{(\alpha)} < \cdots \le 0, \quad \sum_{k=0}^{\infty} g_k^{(\alpha)} = 0,$

(ii) $\{g_k^{(\alpha)}\}_{k\ge 0} \in \mathcal{D}_\alpha.$

For v in (3.1), let $\mathcal{Z}(v)$ denote the zero extension of v over $\mathbb{R}$, i.e.,

$$(\mathcal{Z}(v))(x) = \begin{cases} v(x), & x \in [a,b], \\ 0, & x \in \mathbb{R}\backslash[a,b]. \end{cases}$$

Let $\mathcal{F}(\cdot)$ denote the Fourier transformation. For $\beta > 0$, define

$$\mathfrak{L}^\beta(\mathbb{R}) = \left\{ g : \mathbb{R} \to \mathbb{R} \middle| ||g||_{\mathfrak{L}^\beta(\mathbb{R})} < +\infty \right\},$$

$$||g||_{\mathfrak{L}^\beta(\mathbb{R})} := \left|\left|{}_{-\infty}D_x^\beta g\right|\right|_{L^1(\mathbb{R})} + \left|\left|\mathcal{F}\left({}_{-\infty}D_x^\beta g\right)\right|\right|_{L^1(\mathbb{R})} + \left|\left|{}_xD_\infty^\beta g\right|\right|_{L^1(\mathbb{R})} + \left|\left|\mathcal{F}\left({}_xD_\infty^\beta g\right)\right|\right|_{L^1(\mathbb{R})},$$

where the definitions of the operators, ${}_{-\infty}D_x^\beta$ and ${}_xD_\infty^\beta$, are referred to (1.5) and (1.6), respectively, $||\cdot||_{L^1(\mathbb{R})}$ denotes the L^1 function norm. For a nonnegative integer m, denote

$$C^m(\mathbb{R}) := \left\{ g : \mathbb{R} \to \mathbb{R} \middle| ||g||_{C^m(\mathbb{R})} := \sum_{i=0}^{m} \sup_{x\in\mathbb{R}} |g^{(m)}(x)| < +\infty, \quad g^{(m)} \text{ continuous on } \mathbb{R} \right\},$$

where $g^{(0)} := g$.

Lemma 9 (see [14]) *Assume* v *in* (3.1) *satisfies* $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+1}(\mathbb{R})$ *for any* $\alpha \in (1,2)$*. Then, the spatial discretization* (3.1) *generated by* (4.4) *has 1st-order truncation error, i.e.,*

$$\max_{1\le i\le M} |(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h, \quad \forall \alpha \in (1,2).$$

For positive integer m, positive number β and $i = 1, 2, \ldots, l$, define

$$\mathcal{W}_i^{m,\beta}(\bar{\Omega}) := \left\{ g : \bar{\Omega} \to \mathbb{R} \middle| ||g||_{\mathcal{W}_i^{m,\beta}(\bar{\Omega})} := ||g||_{L_i^{\beta}(\bar{\Omega})} + ||g||_{C_i^m(\bar{\Omega})} < +\infty \right\},$$

$$C_i^m(\bar{\Omega}) = \left\{ g : \bar{\Omega} \to \mathbb{R} \middle| ||g||_{C_i^m(\bar{\Omega})} < +\infty \right\},$$

$$||g||_{L_i^{\beta}(\bar{\Omega})} := \sup_{(x_1,x_2,\ldots,x_l)\in\bar{\Omega}} ||\mathcal{Z}(g(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_l))||_{\mathfrak{L}^{\beta}(\mathbb{R})},$$

$$||g||_{C_i^m(\bar{\Omega})} := \sup_{(x_1,x_2,\ldots,x_l)\in\bar{\Omega}} ||\mathcal{Z}(g(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_l))||_{C^m(\mathbb{R})}.$$

Theorem 10 *Assume*

(i) the assumption (**q**) *holds,*

(ii) for any $t \in [0, T]$, $u(\cdot, t) \in \bigcap_{i=1}^{l} \mathcal{W}_i^{0,\alpha_i+1}(\bar{\Omega})$ *with* $\sup_{t\in[0,T]} \max_{1\le i\le l} ||u(\cdot, t)||_{\mathcal{W}_i^{0,\alpha_i+1}(\bar{\Omega})} \lesssim 1$.

Then, there exists a positive constant c independent of mesh parameters such that whenever $\tau \le c$, the discretized RSFDE (2.4) *with the spatial discretization* (4.4) *is uniquely solvable, unconditionally stable and convergent with*

$$\max_{1\le n\le N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i.$$

Proof From (**i**) of Lemma 8, we see that **S** generated by (4.4) is strictly diagonally dominant with positive diagonal entries, which therefore holds (4.1). By (**ii**) of Lemma 8, $S_1(\{g_k^{(\alpha)}\}_{k\ge0})$, $F_1(\{g_k^{(\alpha)}\}_{k\ge0}) \in \mathcal{D}_\alpha$. As mentioned above, $\mathcal{D}_\alpha$ is a linear space. Hence, (4.2) holds. Applying Lemma 9 to (**ii**) leads to $\max_{0\le n\le N} \max_{J\in\mathbb{K}} |(\partial_i^{\alpha_i} u)(\mathbf{x}_J, t_n) - \delta_i^{\alpha_i} U_J^n| \lesssim h_i$, $i = 1, 2, \ldots, l$. Hence, the results follow from Theorem 7.

Two weighted-shifted Grünwald formulas resulting from [21] can be expressed as

$$\left\{s_k^{(\alpha)}\right\}_{k\ge0} = \mathcal{R}_1\left(\frac{\alpha}{2}\left\{g_k^{(\alpha)}\right\}_{k\ge0} + \frac{2-\alpha}{2}\mathcal{S}_{-1}\left(\left\{g_k^{(\alpha)}\right\}_{k\ge0}\right)\right), \tag{4.5}$$

$$\left\{s_k^{(\alpha)}\right\}_{k\ge0} = \mathcal{R}_1\left(\frac{2+\alpha}{4}\left\{g_k^{(\alpha)}\right\}_{k\ge0} + \frac{2-\alpha}{4}\mathcal{S}_{-2}\left(\left\{g_k^{(\alpha)}\right\}_{k\ge0}\right)\right), \tag{4.6}$$

where $g_k^{(\alpha)}$ $(k \ge 0)$ are given by (4.3).

Lemma 11 (see [21]) (3.1) *generated by both* (4.5) *and* (4.6) *satisfy*

(i) $\mathbf{S} \succ \mathbf{O}$ *for any* $\alpha \in (1, 2)$,

(ii) if $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+2}(\mathbb{R})$, *then* $\max_{1\le i\le M} |(\partial^\alpha v)(a + ih) - \delta^\alpha v(a + ih)| \lesssim h^2$ *for any* $\alpha \in (1, 2)$.

Theorem 12 *Assume*

(i) the assumption (**q**) *holds,*

(ii) for any $t \in [0, T]$, $u(\cdot, t) \in \bigcap_{i=1}^{l} \mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})$ *with* $\sup_{t\in[0,T]} \max_{1\le i\le l} ||u(\cdot, t)||_{\mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})} \lesssim 1$.

Then, there exists a positive constant c independent of mesh parameters such that whenever $\tau \le c$, *the discretized RSFDE* (2.4) *with both* (4.5) *and* (4.6) *are uniquely solvable, unconditionally stable and convergent with* $\max\limits_{1\le n\le N}||\mathbf{e}^n|| \lesssim \tau^2 + \sum\limits_{i=1}^{l} h_i^2$.

Proof (4.1) follows from (**i**) in Lemma 11. (**ii**) in Lemma 8 and the fact that $\mathcal{D}_\alpha$ is a linear space guarantee (4.2). Applying (**ii**) in Lemma 11 to (**ii**) leads to $\max\limits_{0\le n\le N}\max\limits_{J\in\mathbb{K}}|(\partial_i^{\alpha_i}u)(\mathbf{x}_J,t_n)-\delta_i^{\alpha_i}U_J^n| \lesssim h_i^2$, $i=1,2,\ldots,l$. Hence, the results follow from Theorem 7. □

4.2 Verification of Schemes from [6]

In this subsection, we verify the conditions (4.1)–(4.2) for a series of spatial discretizations based on weighted-shifted Lubich difference operators proposed in [6]. Let

$$q_k^{(\alpha)} = -\left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{k} 3^{-j} g_j^{(\alpha)} g_{k-j}^{(\alpha)}, \quad k\ge 0, \quad \alpha\in(1,2),$$

with $g_j^{(\alpha)}(j\ge 0)$ given by (4.3).

A series of spatial discretization resulting from weighting and shifting $\{q_k^{(\alpha)}\}_{k\ge0}$ can be expressed as follows [6]

$$\left\{s_k^{(\alpha)}\right\}_{k\ge0} = \mathscr{L}_p^{(\alpha)} := \theta_p\mathcal{R}_1\left(\left\{q_k^{(\alpha)}\right\}_{k\ge0}\right) + (1-\theta_p)\mathcal{R}_p\left(\left\{q_k^{(\alpha)}\right\}_{k\ge0}\right), \quad \theta_p = \frac{p}{p-1}, \quad p\in\mathbb{N}, \quad |p|\ge 2. \tag{4.7}$$

Weighting and shifting $\mathscr{L}_p^{(\alpha)}$, we obtain some other spatial discretizations as follows [6]

$$\{s_k^{(\alpha)}\}_{k\ge0} = \mathscr{L}_{q,s}^{(\alpha)} := \theta_{q,s}\mathscr{L}_q^{(\alpha)} + (1-\theta_{q,s})\mathscr{L}_s^{(\alpha)}, \quad \theta_{q,s} = \frac{3s+2\alpha}{3(s-q)}, \quad qs<0,\ |q|\ge2,\ |s|\ge2. \tag{4.8}$$

Again, weighting and shifting $\mathscr{L}_{q,s}^{(\alpha)}$, we obtain some other spatial discretizations as follows [6]

$$\left\{s_k^{(\alpha)}\right\}_{k\ge0} = \tilde{\mathscr{L}}_{q,s}^{(\alpha)} := \tilde\theta_{q,s}\mathscr{L}_{2,-2}^{(\alpha)} + (1-\tilde\theta_{q,s})\mathscr{L}_{q,s}^{(\alpha)}, \tag{4.9}$$

where

$$\tilde\theta_{q,s} = \frac{6qs+4\alpha s+4\alpha q+13\alpha}{6qs+4\alpha s+4\alpha q+24}, \quad (q,s)\ne(2,-2), \quad qs<0, \quad |q|\ge2, \quad |s|\ge2.$$

Lemma 13 *All of* $\{s_k^{(\alpha)}\}_{k\ge0}$ *resulting from* (4.7)–(4.9) *satisfy* $\{s_k^{(\alpha)}\}_{k\ge0}\in\mathcal{D}_\alpha$ *for any* $\alpha\in(1,2)$

Proof By (**ii**) in Lemma 8, $\{g_k^{(\alpha)}\}_{k\ge0}\in\mathcal{D}_\alpha$. Denote $c_0 = ||\{g_k^{(\alpha)}\}_{k\ge0}||_{\mathcal{D}_\alpha}$ and $c_1 = 2^{\alpha+1}c_0$. Then, it holds

$$\left|g_k^{(\alpha)}\right| \le \frac{c_0}{(1+k)^{\alpha+1}} = c_0\left(1+\frac{1}{1+k}\right)^{\alpha+1}\frac{1}{(2+k)^{\alpha+1}} \le \frac{c_1}{(2+k)^{\alpha+1}}, \quad k\ge0. \tag{4.10}$$

Denote $c_2 = c_1^2(3/2)^\alpha$. By (4.10),

$$\begin{aligned}
\left|q_k^{(\alpha)}\right| \le \left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{k}\left|g_j^{(\alpha)}\right|\left|g_{k-j}^{(\alpha)}\right| &\le c_2 \sum_{j=0}^{k}\frac{1}{(2+j)^{\alpha+1}(2+k-j)^{\alpha+1}}\\
&\le c_2 \sum_{j=0}^{k}\int_j^{j+1}\frac{1}{(1+x)^{\alpha+1}(2+k-x)^{\alpha+1}}dx\\
&= c_2\left(\int_0^{\frac{k}{2}}+\int_{\frac{k}{2}}^{k+1}\right)\frac{1}{(1+x)^{\alpha+1}(2+k-x)^{\alpha+1}}dx\\
&\le \frac{c_2}{\left(1+\frac{k}{2}\right)^{\alpha+1}}\left[\int_0^{\frac{k}{2}}\frac{1}{(1+x)^{\alpha+1}}dx\right.\\
&\quad\left.+\int_{\frac{k}{2}}^{k+1}\frac{1}{(2+k-x)^{\alpha+1}}dx\right]\\
&= \frac{c_2\left[2-\left(1+\frac{k}{2}\right)^{-\alpha}-\left(2+\frac{k}{2}\right)^{-\alpha}\right]}{\alpha\left(1+\frac{k}{2}\right)^{\alpha+1}}\\
&\le \frac{2^{\alpha+2}c_2}{\alpha(1+k)^{\alpha+1}}, \quad k \ge 0,
\end{aligned}$$

which implies that $||\{q_k^{(\alpha)}\}_{k\ge0}||_{\mathcal{D}_\alpha} < +\infty$. Thus, $\{q_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$. Since $\mathcal{D}_\alpha$ is a linear space, it is easy to check that $\mathcal{R}_m(\{q_k^{(\alpha)}\}_{k\ge0}) \in \mathcal{D}_\alpha$ for any fixed m. Thus, $\{s_k^{(\alpha)}\}_{k\ge0}$ resulting from (4.7)–(4.9) satisfy $\{s_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$, which completes the proof. □

Lemma 14 (see [6])

(i) *Assume* (4.7) *is applied . Then,* $\mathbf{S} \succ \mathbf{O}$. *If in addition* $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+2}(\mathbb{R})$, *then* (3.1) *has 2nd-order truncation error, i.e.,* $\max\limits_{1\le i\le M}|(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^2$ *for any* $\alpha \in (1,2)$.

(ii) *Assume* (4.8) *is applied. Then,* $\mathbf{S} \succ \mathbf{O}$. *If in addition* $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+3}$, *then* (3.1) *has 3rd-order truncation error, i.e.,* $\max\limits_{1\le i\le M}|(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^3$ *for any* $\alpha \in (1,2)$.

(iii) *Assume* (4.9) *is applied. Then,* $\mathbf{S} \succ \mathbf{O}$. *If in addition* $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+4}(\mathbb{R})$, *then* (3.1) *has 4th-order truncation error, i.e.,* $\max\limits_{1\le i\le M}|(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^4$ *for any* $\alpha \in (1,2)$.

Theorem 15 *Assume the assumption* (**q**) *holds.*

(i) *if* $u(\cdot,t) \in \bigcap\limits_{i=1}^{l}\mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})$ *for any* $t \in [0,T]$ *with* $\sup\limits_{t\in[0,T]}\max\limits_{1\le i\le l}||u(\cdot,t)||_{\mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})} \lesssim 1$, *then there exists a mesh-parameters-independent positive constant c such that whenever* $\tau \le c$, *the discretized RSFDE* (2.4) *with spatial discretization* (4.7) *is uniquely solvable, unconditionally stable and convergent with* $\max\limits_{1\le n\le N}||\mathbf{e}^n|| \lesssim \tau^2 + \sum\limits_{i=1}^{l}h_i^2$.

(ii) *if* $u(\cdot,t) \in \bigcap\limits_{i=1}^{l}\mathcal{W}_i^{0,\alpha_i+3}(\bar{\Omega})$ *for any* $t \in [0,T]$ *with* $\sup\limits_{t\in[0,T]}\max\limits_{1\le i\le l}||u(\cdot,t)||_{\mathcal{W}_i^{0,\alpha_i+3}(\bar{\Omega})} \lesssim 1$, *then there exists a mesh-parameters-independent positive constant c such that whenever*

$\tau \le c$, *the discretized RSFDE* (2.4) *with spatial discretization* (4.8) *is uniquely solvable, unconditionally stable and convergent with* $\max_{1\le n\le N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i^3$.

(iii) if $u(\cdot,t) \in \bigcap_{i=1}^{l} \mathcal{W}_i^{0,\alpha_i+4}(\bar{\Omega})$ *for any* $t \in [0,T]$ *with* $\sup_{t\in[0,T]} \max_{1\le i\le l} ||u(\cdot,t)||_{\mathcal{W}_i^{0,\alpha_i+4}(\bar{\Omega})} \lesssim 1$, *then there exists a mesh-parameters-independent positive constant c such that whenever* $\tau \le c$, *the discretized RSFDE* (2.4) *with spatial discretization* (4.9) *is uniquely solvable, unconditionally stable and convergent with* $\max_{1\le n\le N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i^4$.

Proof Similar to proof of Theorems 10, 12, applying Theorem 7 to Lemmas 13–14 leads to the desired results. □

4.3 Verification of Scheme from [8]

In this subsection, we verify conditions (4.1)–(4.2) this subsection, we verify conditionsef-condition2 for the scheme proposed in [8]. Let

$$\omega_k^{(\alpha)} = -\left(\frac{3\alpha-2}{2\alpha}\right)^{\alpha} \sum_{i=0}^{k} \left(\frac{\alpha-2}{3\alpha-2}\right)^{i} g_i^{(\alpha)} g_{k-j}^{(\alpha)}, \quad k=0,1,\ldots, \quad \alpha \in (1,2), \tag{4.11}$$

with $g_k^{(\alpha)}$ given by (4.3). Then, a spatial discretization of form (3.1) resulting from (4.11) is given as follows [8]

$$\left\{s_k^{(\alpha)}\right\}_{k\ge 0} = \mathcal{R}_1\left(\left\{\omega_k^{(\alpha)}\right\}_{k\ge 0}\right). \tag{4.12}$$

Lemma 16 (see [8]) *Let* $\{s_k^{(\alpha)}\}_{k\ge 0}$ *be given by* (4.12). *Then, for any* $\alpha \in (1,2)$, *it holds*

(i) $\mathbf{S} \succ \mathbf{O}$,
(ii) $\{s_k^{(\alpha)}\}_{k\ge 0} \in \mathcal{D}_\alpha$,
(iii) if in addition $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+2}(\mathbb{R})$, *then the discretization* (3.1) *has 2nd-order truncation error, i.e.,* $\max_{1\le i\le M} |(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^2$.

Similar to proof of Theorems 10, 12, applying Theorem 7 to Lemma 16 leads to the desired results.

Theorem 17 *Assume the assumption* (**q**) *holds and* $u(\cdot,t) \in \bigcap_{i=1}^{l} \mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})$ *for any* $t \in [0,T]$ *with* $\sup_{t\in[0,T]} \max_{1\le i\le l} ||u(\cdot,t)||_{\mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})} \lesssim 1$. *Then, there exists a mesh-parameters-independent positive constant c such that whenever* $\tau \le c$, *the discretized RSFDE* (2.4) *with spatial discretization* (4.12) *is uniquely solvable, unconditionally stable and convergent with* $\max_{1\le n\le N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i^2$.

4.4 Verification of Scheme in [4]

In this subsection, we verify conditions (4.1)–(4.2) for the fractional central difference scheme [4], whose coefficients are defined as follows

$$s_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)}, \qquad k \geq 0. \tag{4.13}$$

Lemma 18 (see [4]) *Let* $\{s_k^{(\alpha)}\}_{k\geq 0}$ *be given by* (4.13)*. Then, for any* $\alpha \in (1,2)$*, it holds*

(i) $\mathbf{S} \succ \mathbf{O}$,
(ii) $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$,
(iii) if in addition $\mathcal{Z}(v) \in C^0(\mathbb{R}) \cap \mathfrak{L}^{\alpha+2}(\mathbb{R})$*, then* (3.1) *has 2nd-order truncation error, i.e.,* $\max\limits_{1\leq i\leq M} |(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^2$.

Similar to the proof of Theorems 10, 12, applying Theorem 7 to Lemma 18 leads to the following theorem.

Theorem 19 *Assume the assumption* (**q**) *holds and* $u(\cdot,t) \in \bigcap\limits_{i=1}^{l} \mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})$ *for any* $t \in [0,T]$ *with* $\sup\limits_{t\in[0,T]} \max\limits_{1\leq i\leq l} ||u(\cdot,t)||_{\mathcal{W}_i^{0,\alpha_i+2}(\bar{\Omega})} \lesssim 1$*. Then, there exists a mesh-parameters-independent positive constant c such that whenever* $\tau \leq c$*, the discretized RSFDE* (2.4) *with spatial discretization* (4.13) *is uniquely solvable, unconditionally stable and convergent with* $\max\limits_{1\leq n\leq N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum\limits_{i=1}^{l} h_i^2$.

4.5 Verification of Scheme from [20]

In this subsection, we verify conditions (4.1)–(4.2) for the weighted difference scheme proposed in [20]. Let

$$p_0^{(\alpha)} = -1, \quad p_1^{(\alpha)} = 4 - 2^{3-\alpha}, \quad p_2^{(\alpha)} = -3^{3-\alpha} + 4 \times 2^{3-\alpha} - 6,$$
$$p_k^{(\alpha)} = -(k+1)^{3-\alpha} + 4k^{3-\alpha} - 6(k-1)^{3-\alpha} + 4(k-2)^{3-\alpha} - (k-3)^{3-\alpha}, \quad k \geq 3.$$

Then, a spatial discretization of form (3.1) resulting from $\{p_k^{(\alpha)}\}_{k\geq 0}$ is given as follows

$$\left\{s_k^{(\alpha)}\right\}_{k\geq 0} = \mathcal{R}_1\left(c_\alpha \left\{p_k^{(\alpha)}\right\}_{k\geq 0}\right), \quad c_\alpha = [\Gamma(4-\alpha)]^{-1}, \quad \alpha \in (1,2). \tag{4.14}$$

Lemma 20 (see [20]) *Let* $\{s_k^{(\alpha)}\}_{k\geq 0}$ *be given by* (4.14)*. Then, for any* $\alpha \in (1,2)$*,*

(i) $\mathbf{S} \succ \mathbf{O}$,
(ii) if in addition $\mathcal{Z}(v) \in C^4(\mathbb{R})$*, then* (3.1) *has 2nd-order truncation error, i.e.,* $\max\limits_{1\leq i\leq M} |(\partial^\alpha v)(a+ih) - \delta^\alpha v(a+ih)| \lesssim h^2$.

Proof For $\alpha \in (1,2)$, the following results can be found directly in [20].

$$\sum_{j=0}^{+\infty} p_k^{(\alpha)} = 0, \quad p_1^{(\alpha)} > 0, \quad p_0^{(\alpha)} + p_2^{(\alpha)} < 0, \quad p_k^{(\alpha)} \leq 0, \quad k \geq 3. \tag{4.15}$$

Based on above equalities and inequalities, it is easy to check that **S** resulting from (4.14) is strictly diagonally dominant with positive diagonal entries, which therefore holds $\mathbf{S} \succ \mathbf{O}$. The result (**ii**) can also be found in [20]. ◻

Lemma 21 *Let* $\{s_k^{(\alpha)}\}_{k\geq 0}$ *be given by* (4.14)*. Then,* $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *for any* $\alpha \in (1,2)$.

Proof Define

$$g(x) = x^{3-\alpha}, \quad \Delta_k = g(k+1) - 2g(k) + g(k-1), \quad k \geq 1.$$

By (4.15), $p_k^{(\alpha)} \leq 0$ for $k \geq 3$. Thus,

$$\left| p_k^{(\alpha)} \right| = -p_k^{(\alpha)} = \Delta_k - 2\Delta_{k-1} + \Delta_{k-2}, \quad k \geq 3. \tag{4.16}$$

Using Taylor expansion, it holds

$$\Delta_k = g^{(2)}(k) + w_k, \quad w_k = \int_k^{k+1} \frac{g^{(4)}(\xi)(k+1-\xi)^3}{3!} d\xi + \int_{k-1}^{k} \frac{g^{(4)}(\xi)(1+\xi-k)^3}{3!} d\xi.$$

Moreover,

$$|w_k| \leq \frac{1}{3!}\left(\int_k^{k+1} |g^{(4)}(\xi)| d\xi + \int_{k-1}^{k} |g^{(4)}(\xi)| \right) \leq c_1 (k-1)^{-1-\alpha}, \quad k \geq 2,$$

with $c_1 = \frac{\Gamma(4-\alpha)}{\Gamma(-\alpha)}$. Using Taylor expansion again,

$$g^{(2)}(k) - 2g^{(2)}(k-1) + g^{(2)}(k-2) = g^{(4)}(k-1) + r_k, \quad k \geq 2,$$

with

$$r_k = \int_{k-1}^{k} \frac{g^{(6)}(\xi)(k-\xi)^3}{3!} d\xi + \int_{k-2}^{k-1} \frac{g^{(6)}(\xi)(2+\xi-k)^3}{3!} d\xi, \quad k \geq 2.$$

Besides,

$$|r_k| \leq \frac{1}{3!}\left(\int_{k-1}^{k} |g^{(6)}(\xi)| d\xi + \int_{k-2}^{k-1} |g^{(6)}(\xi)| d\xi \right) \leq c_2 (k-2)^{-3-\alpha}, \quad k \geq 3,$$

with $c_2 = \frac{\Gamma(4-\alpha)}{3!\Gamma(-\alpha-2)}$. Therefore,

$$\begin{aligned}
\left| p_k^{(\alpha)} \right| = \Delta_k - 2\Delta_{k-1} + \Delta_{k-2} &= g^{(2)}(k) - 2g^{(2)}(k-1) + g^{(2)}(k-2) + w_k - 2w_{k-1} + w_{k-2} \\
&= g^{(4)}(k-1) + r_k + w_k - 2w_{k-1} + w_{k-2} \\
&\leq c_2 (k-2)^{-3-\alpha} + c_1 \left[2(k-1)^{-1-\alpha} + 2(k-2)^{-1-\alpha} + (k-3)^{-1-\alpha} \right] \\
&\leq \frac{c_2 + 5c_1}{(k-3)^{1+\alpha}} = \frac{(c_2+5c_1)(1+k)^{1+\alpha}}{(k-3)^{1+\alpha}(1+k)^{1+\alpha}} \leq \frac{5(c_2+5c_1)}{(1+k)^{1+\alpha}}, \quad k \geq 4,
\end{aligned}$$

which implies that $\{p_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$. Moreover, since $\mathcal{D}_\alpha$ is a linear space, $\{s_k^{(\alpha)}\}_{k\geq 0} = \mathcal{R}_1(c_\alpha \{p_k^{(\alpha)}\}_{k\geq 0}) \in \mathcal{D}_\alpha$, which completes the proof. □

Then, similar to the proof of Theorems 10, 12, applying Theorem 7 to Lemmas 20 and 21 leads to the following theorem.

Theorem 22 *Assume the assumption* (**q**) *holds and* $u(\cdot, t) \in \bigcap_{i=1}^{l} C_i^4(\bar{\Omega})$ *for any* $t \in [0, T]$ *with* $\sup_{t\in[0,T]} \max_{1\leq i\leq l} ||u(\cdot,t)||_{C_i^4(\bar{\Omega})} \lesssim 1$. *Then, there exists a mesh-parameters-independent positive constant* c *such that whenever* $\tau \leq c$, *the discretized RSFDE* (2.4) *with spatial discretization* (4.14) *is uniquely solvable, unconditionally stable and convergent with* $\max_{1\leq n\leq N} ||\mathbf{e}^n|| \lesssim \tau^2 + \sum_{i=1}^{l} h_i^2$.

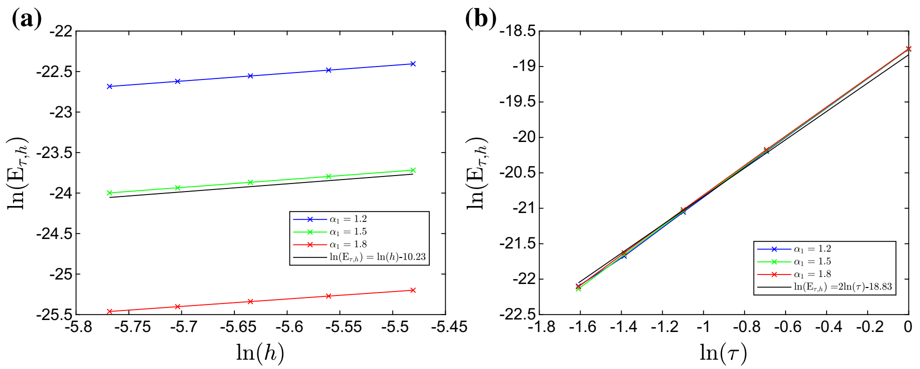

Fig. 1 Verification of convergence rate of Scheme I. **a** $\ln(\mathrm{E}_{\tau,h})$ of Scheme I when $\tau = 1/200$ and $\alpha_2 = 1.8$. **b** $\ln(\mathrm{E}_{\tau,h})$ of Scheme I when $h = 1/500$ and $\alpha_2 = 1.8$

Remark Actually, almost all existing unconditionally stable difference schemes with form (3.1) satisfy the conditions (4.1)–(4.2). We only verify several representative ones among these schemes in this section to demonstrate the conditions (4.1)–(4.2) are general enough.

5 Numerical Experiments

In this subsection, we use one example to examine the convergence order of the discretized RSFDE (2.4) with two different spatial discretizations. Due to limitation of memory of the computer used, we only test the two-dimensional case. Let $h_1 = h_2 = h$. Denote $\mathrm{E}_{\tau,h} = \max\limits_{1\leq n\leq N} ||\mathbf{e}^n||$. Denote by Scheme I , Scheme II, Scheme III and Scheme IV, the Crank–Nicolson scheme (2.4) with $\{s_k^{(\alpha)}\}_{k\geq 0}$ given by (4.4), (4.5), $\mathscr{L}_{2,-2}^{(\alpha)}$ from (4.8) and $\widetilde{\mathscr{L}}_{4,-2}^{(\alpha)}$ from (4.9), respectively.

Example 1 Consider a two-dimensional RSFDE (1.1)–(1.3) with

$$
\begin{aligned}
&u(x_1, x_2, t) = t^2 x_1^6 (1-x_1)^6 x_2^6 (1-x_2)^6, \quad [a_1, b_1] = [a_2, b_2] = [0, 1], \quad T = 1,\\
&d_1(x_1, x_2, t) = (1+t)(1+x_1+x_2+x_1x_2),\\
&d_2(x_1, x_2, t) = \exp(t)(1+x_1+x_2+x_1x_2),\\
&f(x_1, x_2, t) = 2t x_1^6 (1-x_1)^6 x_2^6 (1-x_2)^6\\
&\qquad - t^2 \sigma_{\alpha_1} d_1(x_1, x_2, t) x_2^6 (1-x_2)^6 \sum_{i=6}^{12} \frac{(-1)^i 6! \left[x_1^{i-\alpha_1} + (1-x_1)^{i-\alpha_1}\right]}{(i-6)!(12-i)!}\\
&\qquad - t^2 \sigma_{\alpha_2} d_2(x_1, x_2, t) x_1^6 (1-x_1)^6 \sum_{i=6}^{12} \frac{(-1)^i 6! \left[x_2^{i-\alpha_2} + (1-x_2)^{i-\alpha_2}\right]}{(i-6)!(12-i)!}.
\end{aligned}
$$

We use Example 1 to test Schemes I–IV , the results of which are presented in Figs. 1, 2, 3, 4. Since the ‘slopes’ of red, green and blue ‘lines’ in these figures represent the convergence order of the corresponding schemes, the black lines in Figs. 1, 2, 3, 4 are presented for reference.

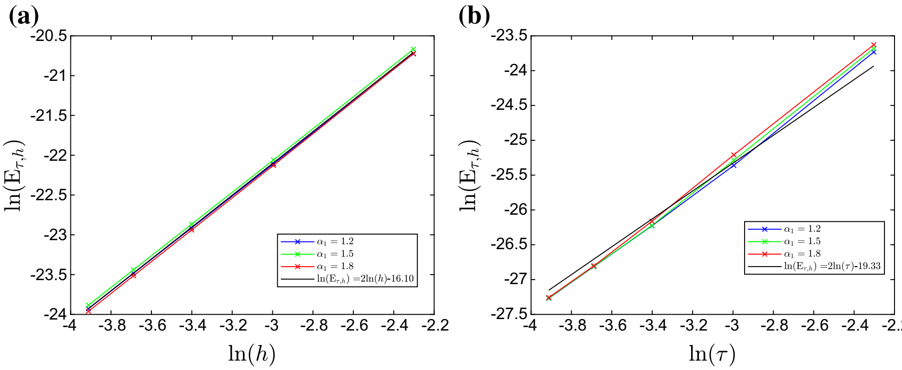

Fig. 2 Verification of convergence rate of Scheme II. **a** $\ln(\mathrm{E}_{\tau,h})$ of Scheme II when $\tau = 1/250$ and $\alpha_2 = 1.8$. **b** $\ln(\mathrm{E}_{\tau,h})$ of Scheme II when $h = 1/250$ and $\alpha_2 = 1.8$

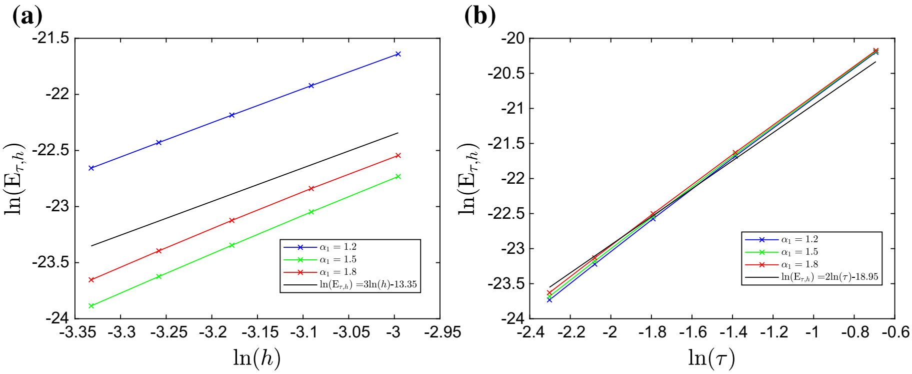

Fig. 3 Verification of convergence rate of Scheme III. **a** $\ln(\mathrm{E}_{\tau,h})$ of Scheme III when $\tau = 1/100$ and $\alpha_2 = 1.8$. **b** $\ln(\mathrm{E}_{\tau,h})$ of Scheme III when $h = 1/100$ and $\alpha_2 = 1.8$

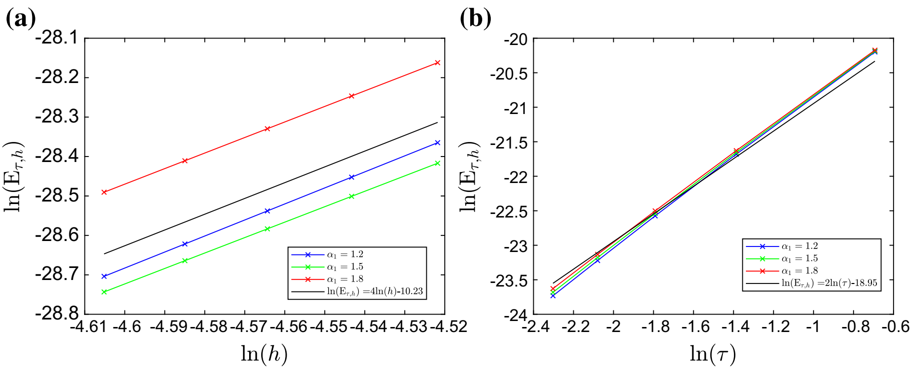

Fig. 4 Verification of convergence rate of Scheme IV. **a** $\ln(\mathrm{E}_{\tau,h})$ of Scheme IV when $\tau = 1/700$ and $\alpha_2 = 1.8$. **b** $\ln(\mathrm{E}_{\tau,h})$ of Scheme IV when $h = 1/50$ and $\alpha_2 = 1.8$

As we see in Fig. 1a, values of $\ln(\mathrm{E}_{\tau,h})$ are distributed close to straight lines with 'slopes' close to 1, which implies that the spatial convergence rate of Scheme I is of $\mathcal{O}(h)$. Also, Fig. 1b shows line-shape distributions of values of $\ln(\mathrm{E}_{\tau,h})$ with 'slopes' close to 2, which implies an $\mathcal{O}(\tau^2)$ temporal convergence rate of Scheme I. To conclude, Fig. 1 shows that Scheme I has a convergence rate of $\mathcal{O}(\tau^2 + h)$, which is in accordance with Theorem 10.

Values of $\ln(\mathrm{E}_{\tau,h})$ in Fig. 2 are distributed like straight lines with 'slopes' close to 2, which demonstrates that the convergence rate of Scheme II is of $\mathcal{O}(\tau^2 + h^2)$, which is in accordance with Theorem 12.

The 'lines' in Fig. 3a, b have 'slopes' close to 3 and 2, respectively, which demonstrates that the convergence rate of Scheme III is of $\mathcal{O}(\tau^2 + h^3)$. Such numerical behavior supports (**ii**) of Theorem 15.

The 'lines' in Fig. 4a, b have 'slopes' close to 4 and 2, respectively, which illustrates $\mathcal{O}(\tau^2 + h^4)$ convergence rate of Scheme IV. This is in accordance with (**iii**) of Theorem 15.

6 Concluding Remarks

In this paper, we have studied a new framework for analysis of Crank–Nicolson scheme with high-order spatial discretization for the RSFDEs and thus a new criterion, (4.1)–(4.2), has been obtained for guaranteeing the unconditional stability and convergence. A series of high-order accurate spatial difference schemes have been proven to satisfy our new criterion. As a result, we obtain a series of temporally 2nd-order and spatially ith-order ($i = 1, 2, 3, 4$) unconditionally stable schemes for the RSFDE (1.1)–(1.3). Our stability and convergence results are new, since, before this paper, the most accurate unconditionally stable scheme for the equation was only temporally 1st-order and spatially 2nd-order. Numerical results have been reported to support our theoretical analysis. We will consider the extension of the analysis to time-space fractional diffusion equations with variable coefficients as our future research works.

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