A separable preconditioner for time-space fractional Caputo-Riesz diffusion equations☆

Xue-lei Lin[b], Michael K. Ng[b], Hai-Wei Sun[a,*]

[a] *Department of Mathematics, University of Macau, Macao*
[b] *Department of Mathematics, Hong Kong Baptist University, Hong Kong*

Abstract

In this paper, we study linear systems arising from time-space fractional Caputo-Riesz diffusion equations with time-dependent diffusion coefficients. The coefficient matrix is a summation of a block-lower-triangular-Toeplitz matrix (temporal component) and a block-diagonal-with-diagonal-times-Toeplitz-block matrix (spatial component). The main aim of this paper is to propose separable preconditioners for solving these linear systems, where a block ϵ-circulant preconditioner is used for the temporal component, while a block diagonal approximation is used for the spatial variable. The resulting preconditioner can be block-diagonalized in the temporal domain. Furthermore, the fast solvers can be employed to solve smaller linear systems in the spatial domain. Theoretically, we show that if the diffusion coefficient (temporal-dependent or spatial-dependent only) function is smooth enough, the singular values of the preconditioned matrix are bounded independent of discretization parameters. Numerical examples are tested to show the performance of proposed preconditioner.

Keywords: Block lower triangular; Toeplitz-like matrix; Diagonalization; Separable; Block ϵ-circulant preconditioner; Time-space fractional diffusion equations.

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1. Introduction

Consider an initial-boundary value problem of the time-space fractional diffusion equation (TSFDE) (see [5])

$$ {}_0^C D_t^{\alpha} u(x,t) = d(x,t)\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t), \qquad (x,t) \in (a,b)\times(0,T], \tag{1.1} $$

$$ u(a,t) = u(b,t) = 0, \qquad t \in (0,T], \tag{1.2} $$

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*The Corresponding Author.

Email address: HSun@umac.mo (Hai-Wei Sun)

$$u(x,0)=\psi(x), \qquad x\in[a,b], \tag{1.3}$$

where $d(x,t)\geq 0$, $u(x,t)$ is unknown to be solved, $f(x,t)$ is source term, $\psi(x)$ is initial condition, ${}_0^C D_t^{\alpha} u$ is the Caputo's derivative of order $\alpha(0<\alpha<1)$ with respect to t defined by

$${}_0^C D_t^{\alpha} u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha}ds, \tag{1.4}$$

$\Gamma(\cdot)$ denotes the gamma function, $\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}}$ is the Riesz fractional derivative of order $\beta(1<\beta<2)$ with respect to x defined by

$$\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}}:=\sigma_{\beta}\left({}_a D_x^{\beta}+\ {}_x D_b^{\beta}\right)u(x,t), \quad (x,t)\in(a,b)\times(0,T], \quad \sigma_{\beta}=-\frac{1}{2\cos(\frac{\pi\beta}{2})}>0,$$

with the left and the right sided Riemann-Liouville derivatives, ${}_a D_x^{\beta} u(x,t)$ and ${}_x D_b^{\beta} u(x,t)$ being defined by

$${}_a D_x^{\beta} u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}}d\xi,$$

and

$${}_x D_b^{\beta} u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,t)}{(\xi-x)^{\beta-1}}d\xi,$$

respectively.

Fractional differential equations are a class of differential equations where the integer-order-derivative terms are replaced by fractional-order derivative. There are several non-equivalent definitions of fractional derivative; see [7, 12, 16]. The Caputo's fractional derivative is often used for time-fractional derivative. The Riemann-Liouville derivatives and the Riesz fractional derivative are often used as space-fractional derivatives. Since closed-form analytical solutions of fractional differential equations are often unavailable especially in the existence of variable coefficients, a lot of useful numerical approximations has been developed for these fractional derivatives; see [1, 3, 4, 6, 8, 9, 11, 15, 18, 19, 21, 23, 25]. Nevertheless, as the fractional differential operators are nonlocal, the discretization of the Caputo's derivative is history-dependent and the discretization of the Riesz's derivative lead to a dense spatial matrix. Therefore, direct solver for the linear systems arising from discretization of TSFDEs requires very high computational complexity when the grid is dense. This motivates us to develop fast solvers for linear systems arising from TSFDEs.

For uniform-grid discretization of the TSFDEs with non-constant coefficients, the resulting coefficient matrix is block lower triangular Toeplitz-like with Toeplitz-like blocks. For such linear systems, the well-known block forward substitution method with Gaussian elimination inner solver requires $\mathcal{O}(MN^2+M^3N)$ operations and $\mathcal{O}(MN+M^2)$ storage, provided that

N is the number of blocks in the coefficient matrix and M is the order of each block. The computational cost is quite expensive compared with the number of the unknowns (MN). Recently, Zhao, Jin and Lin [26] proposed to use a combination of time-stepping method and preconditioned generalized minimum residual (PGMRES) method with a banded preconditioner to solve space-time fractional advection-diffusion equations. Their method can also be regarded as the block forward substitution method with PGMRES inner solver, which can be extended to solve linear systems arising from TSFDEs. The computational cost of such combination method is of $\mathcal{O}(MN^2 + NM \log M)$ and the storage is of $\mathcal{O}(MN)$. On the other hand, since the coefficient matrices arising from TSFDEs are block lower triangular Toeplitz-like with Toeplitz-like blocks matrices, Krylov subspace iterative methods can be employed for solving such linear systems. The corresponding matrix-vector multiplication can be performed efficiently using Fast Fourier Transforms (FFTs). The storage requirement is of $\mathcal{O}(MN)$ and the computational cost is of $\mathcal{O}(MN \log(MN))$ operations per iteration. Nevertheless, the coefficient matrices are ill-conditioned. Therefore, the Krylov subspace method converges very slowly without preconditioners.

The main objective of this paper is to develop separable preconditioners for these block lower triangular Toeplitz-like with Toeplitz-like blocks coefficient matrices arising from TSFDEs. Our idea is to use a block ϵ-circulant approximation (see [10]) in the temporal component of the coefficient matrix, and a block diagonal approximation in the spatial component of the coefficient matrix. The resulting preconditioner can be block-diagonalized in the temporal domain and hence fast solvers are employed to solve the smaller linear systems (blocks) in the spatial domain. Theoretically, we show that if the diffusion coefficient function (temporal-dependent $d(x,t) = d(t)$ or spatial-dependent $d(x,t) = d(x)$ only) is smooth enough, the singular values of the preconditioned matrix are bounded above and below by constants which are independent of discretization parameters. Thus, when the Krylov subspace method is employed to solve these preconditioned systems, it converges very quickly. Because the temporal component of the preconditioner is diagonalizable, the preconditioner can be decomposed into several smaller matrices. Fast solvers such as multigrid methods are employed to solving these smaller linear systems in the spatial domain. Therefore, the proposed preconditioner is implemented efficiently. Numerical examples are reported to demonstrate the performance of the proposed preconditioner is better than other approaches for solving TSFDEs.

The rest of this paper is organized as follows. In Section 2, we present the discretized form of the TSFDE (1.1)–(1.3). In Section 3, we construct the separable preconditioner, analyze the singular values of this preconditioned matrix. In Section 4, we introduce the multigrid method for solving linear systems arising from the proposed preconditioning strategy. In Section 5, experimental results are presented to show the performance of the proposed preconditioner. Finally, some concluding remarks are given in Section 6.

2. Discretization of TSFDEs

In this section, we present a uniform-grid discretization of TSFDE (1.1)–(1.3) and the discretized linear system. For positive integers M and N, let $\tau = T/N$ and $h = (b-a)/(M+1)$. Define the temporal-grid and the spatial-grid respectively by

$$\{t_n | t_n = n\tau,\ 0 \le n \le N\} \qquad \text{and} \qquad \{x_i | x_i = a + ih,\ 0 \le i \le M+1\}.$$

Also, define the grid functions

$$\{u_{in} = u(x_i, t_n) | 0 \le i \le M+1, \quad 0 \le n \le N\},\ \{f_{in} = f(x_i, t_n) | 0 \le i \le M+1, \quad 0 \le n \le N\},$$
$$\{d_{in} = d(x_i, t_n) | 0 \le i \le M+1, \quad 0 \le n \le N\}, \quad \{\psi_i = \psi(x_i) | 0 \le i \le M+1\}.$$

We employ the $L1$ formula (see [8]) to approximate ${}_0^C D_t^\alpha$ as follows

$${}_0^C D_t^\alpha u|_{(x,t)=(x_i,t_n)} \approx D_\tau^\alpha u_{in} := \tau^{-\alpha} \sum_{k=1}^{n} w_{n-k}^{(\alpha)} u_{ik} + w^{(n,\alpha)} u_{i0}, \quad 1 \le n \le N, \quad 0 \le i \le M+1, \tag{2.1}$$

where

$$w_k^{(\alpha)} = \begin{cases} [\Gamma(2-\alpha)]^{-1}, & k = 0, \\ [\Gamma(2-\alpha)]^{-1}[(k+1)^{1-\alpha} - 2k^{1-\alpha} + (k-1)^{1-\alpha}], & 1 \le k \le N-1, \end{cases} \tag{2.2}$$

$$w^{(k,\alpha)} = [(k-1)^{1-\alpha} - k^{1-\alpha}][\tau^\alpha \Gamma(2-\alpha)]^{-1}, \qquad 1 \le k \le N. \tag{2.3}$$

For the approximation of Riemann-Liouville derivatives, we refer to the shifted Grünwald formula proposed in [11],

$${}_a D_x^\beta u|_{(x,t)=(x_i,t_n)} \approx -\frac{1}{h^\beta} \sum_{j=1}^{i+1} g_{i-j+1}^{(\beta)} u(x_j, t_n), \qquad 1 \le i \le M, \quad 0 \le n \le N, \tag{2.4}$$

$${}_x D_b^\beta u|_{(x,t)=(x_i,t_n)} \approx -\frac{1}{h^\beta} \sum_{j=i-1}^{M} g_{j-i+1}^{(\beta)} u(x_j, t_n), \quad 1 \le i \le M, \quad 0 \le n \le N, \tag{2.5}$$

where

$$g_0^{(\beta)} = -1, \quad g_{k+1}^{(\beta)} = \left(1 - \frac{\beta+1}{k+1}\right) g_k^{(\beta)}, \quad k = 0, 1, 2, ... \tag{2.6}$$

Then, (2.4)–(2.5) induce an approximation of Riesz fractional derivative as follows

$$\left.\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}\right|_{(x,t)=(x_i,t_n)} \approx -\frac{1}{h^\beta} \sum_{j=1}^{M} s_{|i-j|}^{(\beta)} u(x_j, t_n), \quad 1 \le i \le M, \quad 0 \le n \le N, \tag{2.7}$$

with

$$s_{|i-j|}^{(\beta)}=\begin{cases}2\sigma_\beta g_1^{(\beta)}, & i=j,\\ \sigma_\beta(g_2^{(\beta)}+g_0^{(\beta)}), & |i-j|=1,\\ \sigma_\beta g_{|i-j|+1}^{(\beta)}, & |i-j|>1.\end{cases} \tag{2.8}$$

Remark: Actually, the $L2\text{-}1_\sigma$ formula proposed in [1] can also be incorporated into the form of (2.1) by some variation. Moreover, the discretization schemes proposed in [17], [18], and [19] lead to the same form as that of (2.7). Our proposed preconditioner in the following also works well for these discretization schemes and the theoretical analysis is similar to the analysis presented in Section 3.

Applying (2.1), (2.4), and (2.5) to the TSFDE (1.1)–(1.3), we obtain a discretized TSFDE as follows

$$D_\tau^\alpha u_{in}=-\frac{d_{in}}{h^\beta}\sum_{j=1}^{M}s_{|i-j|}^{(\beta)}u(x_j,t_n)+f_{in}, \qquad 1\le i\le M,\quad 1\le n\le N, \tag{2.9}$$

$$u_{0,n}=u_{M+1,n}=0, \qquad 1\le n\le N, \tag{2.10}$$

$$u_{i,0}=\psi_i, \qquad 0\le i\le M+1. \tag{2.11}$$

The equations (2.9)–(2.11) are actually equivalent to the following linear system

$$\mathbf{A}\mathbf{u}=\mathbf{f}, \tag{2.12}$$

where

$$\begin{aligned}
&\mathbf{f}=(\mathbf{f}_1,\mathbf{f}_2,...,\mathbf{f}_N)^{\mathrm{T}}-\mathbf{v}_\alpha\otimes\Psi, && \mathbf{f}_n=(f_{1n},f_{2n},...,f_{Mn}), && 1\le n\le N,\\
&\mathbf{v}_\alpha=(w^{(1,\alpha)},w^{(2,\alpha)},...,w^{(N,\alpha)})^{\mathrm{T}}, && \Psi=(\psi_1,\psi_2,...,\psi_M)^{\mathrm{T}},\\
&\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_N)^{\mathrm{T}}, && \mathbf{u}_n=(u_{1n},u_{2n},...,u_{Mn}), && 1\le n\le N,\\
&\mathbf{A}=\mathbf{T}\otimes\mathbf{I}_M+\mathbf{D}(\mathbf{I}_N\otimes\mathbf{S}), && \mathbf{D}=\mathrm{diag}(\mathbf{D}_1,\mathbf{D}_2,...,\mathbf{D}_N),\\
&\mathbf{D}_n=\mathrm{diag}(d_{1n},d_{2n},...,d_{Mn}), && 1\le n\le N,
\end{aligned}$$

$$\mathbf{T}=\frac{1}{\tau^\alpha}\begin{bmatrix} w_0^{(\alpha)} & & & & \\ w_1^{(\alpha)} & w_0^{(\alpha)} & & & \\ w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & & \\ \vdots & \ddots & \ddots & \ddots & \\ w_{N-1}^{(\alpha)} & \dots & w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)}\end{bmatrix},\quad \mathbf{S}=\frac{1}{h^\beta}\begin{bmatrix} s_0^{(\beta)} & s_1^{(\beta)} & \dots & s_{M-2}^{(\beta)} & s_{M-1}^{(\beta)}\\ s_1^{(\beta)} & s_0^{(\beta)} & s_1^{(\beta)} & \dots & s_{M-2}^{(\beta)}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ s_{M-2}^{(\beta)} & \dots & s_1^{(\beta)} & s_0^{(\beta)} & s_1^{(\beta)}\\ s_{M-1}^{(\beta)} & s_{M-2}^{(\beta)} & \dots & s_1^{(\beta)} & s_0^{(\beta)}\end{bmatrix},$$

$\mathbf{I}_k$ denotes $k\times k$ identity matrix, "$\otimes$" denotes the Kronecker product, $\mathbf{u}$ is the unknown to be solved. Using the same technique presented in [5], one can prove the stability and $\mathcal{O}(\tau^{2-\alpha}+h)$ convergence order of the scheme (2.9)–(2.11) in the sense of infinity norm. The matrix $\mathbf{T}$ is a lower triangular Toeplitz matrix. The matrix $\mathbf{S}$ is a symmetric Toeplitz matrix. Here $\mathbf{A}$ is a block lower triangular Toeplitz-like with Toeplitz-like blocks matrix. It is easy to see that matrix-vector multiplication of $\mathbf{A}$ can be fast computed with only $\mathcal{O}(MN\log(MN))$ operation cost and $\mathcal{O}(MN)$ storage by the FFTs and property of Kronecker product.

3. Separable Preconditioner

In this section, we construct a separable preconditioner for the linear system in (2.12) and analyze the singular values of the preconditioned matrix. Let

$$\bar{\mathbf{P}} = \mathbf{T} \otimes \mathbf{I}_M + \mathbf{I}_N \otimes (\bar{\mathbf{D}}\mathbf{S}), \qquad \bar{\mathbf{D}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{D}_n.$$

It is obvious that $\bar{\mathbf{P}}$ can simply derive from replacing $\mathbf{D}$ in $\mathbf{A}$ by $\mathbf{I}_N \otimes \bar{\mathbf{D}}$. Thus, when $d(x,t)$ is smooth enough with respect to the time variable t, $\bar{\mathbf{P}}^{-1}\mathbf{A}$ is close to the identity. According to [10], $\bar{\mathbf{P}}$ can be approximated by a block ϵ-circulant matrix $\mathbf{P}_\epsilon$ as follows

$$\mathbf{P}_\epsilon = \bar{\mathbf{P}} + \epsilon\widehat{\mathbf{T}} \otimes \mathbf{I}_M,$$

with $\epsilon \in (0,1]$ being a small number and

$$\widehat{\mathbf{T}} = \frac{1}{\tau^\alpha}\begin{bmatrix} 0 & w_{N-1}^{(\alpha)} & \dots & w_2^{(\alpha)} & w_1^{(\alpha)} \\ & 0 & w_{N-1}^{(\alpha)} & \ddots & w_2^{(\alpha)} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & w_{N-1}^{(\alpha)} \\ & & & & 0 \end{bmatrix}. \tag{3.1}$$

The advantage of $\mathbf{P}_\epsilon$ is that it can be block diagonalized in the following way (see [10])

$$\mathbf{P}_\epsilon = \left[(\mathbf{\Phi}^{-1}\mathbf{F}_N^*) \otimes \mathbf{I}_M\right] \mathrm{diag}\left(\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \dots, \mathbf{\Lambda}_{N-1}\right) \left[(\mathbf{F}_N\Phi) \otimes \mathbf{I}_M\right], \tag{3.2}$$

where $\Phi = \mathrm{diag}(\phi^0, \phi^1, ..., \phi^{N-1})$ with $\phi = \sqrt[N]{\epsilon}$, $\mathbf{F}_N$ is the $N \times N$ Fourier transformation matrix such that

$$\mathbf{F}_N = \frac{1}{\sqrt{N}}\left[\exp(2\pi\mathbf{i}ij/N)\right]_{i,j=0}^{N-1}, \qquad \mathbf{i} \equiv \sqrt{-1},$$

and

$$\mathbf{\Lambda}_k = \lambda_k\mathbf{I}_M + \bar{\mathbf{D}}\mathbf{S}, \qquad \lambda_k = \frac{1}{\tau^\alpha}\sum_{j=0}^{N-1}\phi^j w_j^{(\alpha)}\exp(2\pi\mathbf{i}kj/N), \quad 0 \le k \le N-1.$$

By (3.2), the matrix-vector multiplication, $\mathbf{P}_\epsilon^{-1}\mathbf{z}$ for a general vector $\mathbf{z}$ can be computed as follows

$$\mathbf{z}_1 = [(\mathbf{F}_N\Phi)\otimes \mathbf{I}_M]\,\mathbf{z}, \tag{3.3}$$

$$\mathrm{diag}\,(\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_{N-1})\,\mathbf{z}_2 = \mathbf{z}_1, \tag{3.4}$$

$$\mathbf{P}_\epsilon^{-1}\mathbf{z} = \left[(\Phi^{-1}\mathbf{F}_N^*)\otimes \mathbf{I}_M\right]\mathbf{z}_2. \tag{3.5}$$

The computation of (3.3) and (3.5) only require $\mathcal{O}(MN\log(MN))$ operations and $\mathcal{O}(MN)$ storage; see [14]. Thus, compared with $\bar{\mathbf{P}}^{-1}\mathbf{z}$ and $\mathbf{A}^{-1}\mathbf{z}$, $\mathbf{P}_\epsilon^{-1}\mathbf{z}$ is much easier to implement since the large-scale problem is converted into N small-scale spatial problems in (3.4). This explains why the block-ϵ-circulant approximation is efficient. The computation of (3.4) will be discussed in Section 4.

3.1. Properties of the Separable Preconditioner

An essential property of $\mathbf{P}_\epsilon$ as a preconditioner is its invertibility. Before proving the invertibility, we firstly introduce several lemmas, which will come into use afterwards.

Lemma 1.
For any $\alpha \in (0,1)$, it holds

(i) $w_0^{(\alpha)} > 0$ *and* $w_k^{(\alpha)} < 0, \quad k \geq 1;$

(ii) *for any* $m \geq 1$, $w_0^{(\alpha)} - \sum\limits_{k=1}^{m} |w_k^{(\alpha)}| > \frac{c_\alpha}{(m+\frac{1}{2})^\alpha} > 0$, *where* $c_\alpha = \frac{(1-\alpha)}{\Gamma(2-\alpha)}$.

Proof: Notice that

$$\begin{aligned} w_0^{(\alpha)} &= \frac{1}{\Gamma(2-\alpha)} > 0, \\ w_k^{(\alpha)} &= [\Gamma(2-\alpha)]^{-1}[(k+1)^{1-\alpha} - 2k^{1-\alpha} + (k-1)^{1-\alpha}] \\ &= \frac{(1-\alpha)}{\Gamma(2-\alpha)}\left(\int_k^{k+1} x^{-\alpha}dx - \int_{k-1}^{k} x^{-\alpha}dx\right) < 0, \quad k \geq 1, \end{aligned} \tag{3.6}$$

which proves (i).

By (3.6),

$$\begin{aligned} w_0^{(\alpha)} - \sum_{k=1}^{m} |w_k^{(\alpha)}| = w_0^{(\alpha)} + \sum_{k=1}^{m} w_k^{(\alpha)} &= \frac{1}{\Gamma(2-\alpha)} + \frac{(1-\alpha)}{\Gamma(2-\alpha)}\sum_{k=1}^{m}\left(\int_k^{k+1} x^{-\alpha}dx - \int_{k-1}^{k} x^{-\alpha}dx\right) \\ &= \frac{(1-\alpha)}{\Gamma(2-\alpha)}\int_m^{m+1} x^{-\alpha}dx. \end{aligned} \tag{3.7}$$

Let $v(x) = x^{-\alpha}$. Denote $x_* = m + \frac{1}{2}$. Using Taylor expansion,

$$v(x) = v(x_*) + (x - x_*)v'(x_*) + \frac{1}{2}(x - x_*)^2 v''(\xi_x), \quad x \in [m, m+1],$$

where $\xi_x \in [m, m+1]$ is a number depending on $x \in [m, m+1]$. Hence,

$$\begin{aligned}\int_m^{m+1} v(x)dx &= v(x_*) + v'(x_*) \int_m^{m+1} (x - x_*)dx + \frac{1}{2} \int_m^{m+1} (x - x_*)^2 v''(\xi_x)dx \\ &= v(x_*) + \frac{\alpha(1+\alpha)}{2} \int_m^{m+1} (x - x_*)^2 \xi_x^{-\alpha-2} dx > v(x_*). \end{aligned} \tag{3.8}$$

The result (ii) follows from (3.7) and (3.8). □

Lemma 2. (see [13]) *For any $x \geq 1$, the gamma function holds*

$$\sqrt{\frac{2\pi}{e}} \left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp B_1(x) < \Gamma(x+1) < \sqrt{\frac{2\pi}{e}} \left(\frac{x+1}{e}\right)^{x+\frac{1}{2}} \exp B_2(x),$$

where e denotes the Euler's number,

$$B_1(x) = \frac{1}{12x} - \frac{1}{12x^2} + \frac{29}{360x^3} - \frac{3}{40x^4} + \frac{17}{252x^5} - \frac{5}{84x^6}, \quad B_2(x) = B_1(x) + \frac{89}{1680x^7}.$$

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ complex matrices. For any $\mathbf{C} = [c_{i,j}]_{i,j=1}^m \in \mathbb{C}^{m\times m}$, denote

$$\mathtt{dd}_i(\mathbf{C}) = |c_{i,i}| - \sum_{j=1, j\neq i}^{m} |c_{i,j}|, \quad 1 \leq i \leq m.$$

Especially, it is well known that if $\mathtt{dd}_i(\mathbf{C}) > 0$ holds for all $1 \leq i \leq m$, then $\mathbf{C}$ is said to be strictly diagonally dominant (SDD).

Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and the maximal eigenvalue of a Hermitian matrix.

Lemma 3.
For any $\beta \in (1, 2)$, it holds

(i) $g_1^{(\beta)} > 0, \;\; g_0^{(\beta)} < g_2^{(\beta)} < g_3^{(\beta)} < \cdots \leq 0, \quad \sum_{k=0}^{\infty} g_k^{(\beta)} = 0$;

(ii) $g_k^{(\beta)} = \frac{-\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)}$, *for any* $k \geq 0$;

(iii) $\mathbf{S}$ *is Hermitian positive definite and* $\lambda_{\min}(\mathbf{S}) \geq c_\beta > 0$ *with* $c_\beta = \frac{2^{\beta+1}\sigma_\beta \iota_\beta}{\beta 3^\beta (b-a)^\beta}$ *and*

$$\iota_\beta = \min\left\{1, \frac{(3-\beta)(2-\beta)(\beta-1)\beta 5^{1+\beta}}{24}, \frac{1}{\Gamma(-\beta)} \exp\left(\frac{-1}{1474719}\right)\left(\frac{6e}{5-\beta}\right)^{1+\beta}\right\}.$$

Proof: The results (i) and (ii) can be found directly in [22]. We only need to prove (iii).

Denote $c_* = \inf_{k\geq 0} |g_k^{(\beta)}|(1+k)^{1+\beta}$. Then, it is easy to see that

$$|g_k^{(\beta)}| \geq \frac{c_*}{(1+k)^{1+\beta}}, \quad k \geq 0. \tag{3.9}$$

By (2.8), (i) and (3.9), it holds for any $1 \leq i \leq M$ that

$$\begin{aligned}
\mathtt{dd}_i(\mathbf{S}) &= \frac{\sigma_\beta}{h^\beta}\left(|2g_1^{(\beta)}| - 2|g_0^{(\beta)} + g_2^{(\beta)}| - \sum_{j=i-1, |j|>1}^{i-M} |g_{|j|+1}^{(\beta)}|\right) \\
&= \frac{\sigma_\beta}{h^\beta}\left(\sum_{j=0}^{i} g_j^{(\beta)} + \sum_{j=0}^{M-i+1} g_j^{(\beta)}\right) \\
&> \frac{2\sigma_\beta}{h^\beta}\sum_{j=0}^{M} g_j^{(\beta)} \\
&= \frac{2\sigma_\beta}{h^\beta}\sum_{j=M+1}^{\infty} |g_j^{(\beta)}| \\
&\geq \frac{2\sigma_\beta}{h^\beta}\sum_{j=M+1}^{\infty} \frac{c_*}{(1+j)^{1+\beta}} \geq \frac{2\sigma_\beta}{h^\beta}\sum_{j=M+1}^{\infty}\int_{j+1}^{j+2} \frac{c_*}{x^{1+\beta}}dx \geq \frac{2^{\beta+1}\sigma_\beta c_*}{\beta 3^\beta (b-a)^\beta}.
\end{aligned} \tag{3.10}$$

Next, we are going to estimate c_*. From the definition of c_*, we see that $c_* = \min\{c_{*,1}, c_{*,2}\}$ with $c_{*,1} = \min_{0\leq k\leq 4} |g_k^{(\beta)}|(1+k)^{1+\beta}$ and $c_{*,2} = \inf_{k\geq 5} |g_k^{(\beta)}|(1+k)^{1+\beta}$. Using (2.6), we have

$$c_{*,1} = \min\left\{1, \frac{(3-\beta)(2-\beta)(\beta-1)\beta 5^{1+\beta}}{24}\right\}. \tag{3.11}$$

Denote $\tilde{k} = k - 1 - \beta$ for $k \geq 5$. Then, by (ii) and Lemma 2, we obtain

$$|g_k^{(\beta)}| = \frac{\Gamma(\tilde{k}+1)}{\Gamma(-\beta)\Gamma(k+1)} > \left[\frac{\exp(B_1(\tilde{k}) - B_2(k))}{\Gamma(-\beta)}\right]\left(\frac{\tilde{k}+1}{e}\right)^{\tilde{k}+\frac{1}{2}}\left(\frac{e}{k+1}\right)^{k+\frac{1}{2}}, \quad k \geq 5, \tag{3.12}$$

where $B_1(\cdot)$ and $B_2(\cdot)$ are defined in Lemma 2. By checking derivative of $B_1(x)$, it is easy to see that $B_1(x)$ monotonically decreases on the interval $x \in [2, +\infty)$. Therefore,

$$B_1(\tilde{k}) - B_2(k) = B_1(\tilde{k}) - B_1(k) - \frac{89}{1680k^7} \geq \frac{-89}{1680k^7} > \frac{-1}{1474719}, \quad k \geq 5. \tag{3.13}$$

On the other hand,

$$
\begin{aligned}
\left(\frac{\tilde{k}+1}{e}\right)^{\tilde{k}+\frac{1}{2}}\left(\frac{e}{k+1}\right)^{k+\frac{1}{2}} = \frac{e^{1+\beta}(\tilde{k}+1)^{\tilde{k}+\frac{1}{2}}}{(k+1)^{k+\frac{1}{2}}} &\geq \frac{e^{1+\beta}(\tilde{k}+1)^{\tilde{k}+\frac{1}{2}}}{(\tilde{k}+1)^{k+\frac{1}{2}}} \\
&= \left[\left(\frac{e}{1+k}\right)\left(1+\frac{1+\beta}{k-\beta}\right)\right]^{1+\beta} \\
&\geq \left[\left(\frac{e}{1+k}\right)\left(1+\frac{1+\beta}{5-\beta}\right)\right]^{1+\beta} \\
&= \left(\frac{6e}{5-\beta}\right)^{1+\beta}\left(\frac{1}{1+k}\right)^{1+\beta}, \quad k \geq 5. \quad (3.14)
\end{aligned}
$$

By (3.12), (3.13) and (3.14), we obtain, $|g_k^{(\beta)}| > \frac{1}{\Gamma(-\beta)}\exp\left(\frac{-1}{1474719}\right)\left(\frac{6e}{5-\beta}\right)^{1+\beta}\left(\frac{1}{1+k}\right)^{1+\beta}$ for $k \geq 5$, which implies that $c_{*,2} \geq \frac{1}{\Gamma(-\beta)}\exp\left(\frac{-1}{1474719}\right)\left(\frac{6e}{5-\beta}\right)^{1+\beta}$. This together with (3.11) induces that

$$
\begin{aligned}
c_* = \min\{c_{*,1}, c_{*,2}\} &\geq \min\left\{1, \frac{(3-\beta)(2-\beta)(\beta-1)\beta 5^{1+\beta}}{24}, \frac{1}{\Gamma(-\beta)}\exp\left(\frac{-1}{1474719}\right)\left(\frac{6e}{5-\beta}\right)^{1+\beta}\right\} \\
&= \iota_\beta.
\end{aligned}
$$

Hence, from (3.10), we have $\min\limits_{1\leq i\leq M} \mathtt{dd}_i(\mathbf{S}) \geq \frac{2\sigma_\beta \iota_\beta}{\beta(b-a)^\beta} = c_\beta > 0$, which implies $\mathbf{S}$ is SDD. Note that $\mathbf{S}$ is also a Hermitian matrix with positive diagonal entries. Hence, by Gershgorin circle theorem, $\mathbf{S}$ is Hermitian positive definite and $\lambda_{\min}(\mathbf{S}) \geq c_\beta$. The proof is complete. □

Let Re(·) denote real part of a complex number.

Theorem 4.
For any positive integers M, N, any $\alpha \in (0,1)$ and any $\beta \in (1,2)$, it holds

(i) $\mathbf{A}$ *is invertible;*

(ii) $\mathbf{P}_\epsilon$ *is invertible for any $\epsilon \in (0,1]$.*

Proof: Since $\mathbf{A}$ is a block lower triangular matrix, to prove its invertibility is equivalent to prove the invertibility of its block diagonal. Indeed, from (3.10) and nonnegativity of $\mathbf{D}$, we see that $\tau^{-\alpha}w_0^{(\alpha)}\mathbf{I}_{MN} + \mathbf{D}(\mathbf{I}_N \otimes \mathbf{S})$ is SDD and thus invertible, which proves (i).

By (3.2), to show the invertibility of $\mathbf{P}_\epsilon$ is equivalent to show the invertibility of $\mathbf{\Lambda}_k = \lambda_k \mathbf{I}_M + \bar{\mathbf{D}}\mathbf{S}$ for all $0 \leq k \leq N-1$. Note that $\bar{\mathbf{D}}$ is nonnegative and $\mathbf{S}$ is SDD with positive diagonal entries. Hence, if λ_k has positive real part, $\mathbf{\Lambda}_k$ will be also SDD and thus invertible. Indeed, by Lemma 1,

$$
\min_{0\leq k\leq N-1} \mathrm{Re}(\lambda_k) = \min_{0\leq k\leq N-1} \frac{1}{\tau^\alpha}\left(w_0^{(\alpha)} - \sum_{j=1}^{N-1}\phi^j w_j^{(\alpha)}\cos(2\pi kj/N)\right)
$$

$$\geq \frac{1}{\tau^{\alpha}}\left(w_0^{(\alpha)} - \sum_{j=1}^{N-1} |w_j^{(\alpha)}|\right) > 0. \tag{3.15}$$

Thus, we conclude that $\mathbf{\Lambda}_k$ is invertible for all $0 \leq k \leq N-1$, which completes the proof. □

3.2. Singular Values of $\mathbf{P}_\epsilon^{-1}\mathbf{A}$

In this subsection, we study singular values of $\mathbf{P}_\epsilon^{-1}\mathbf{A}$. At first, several notations and lemmas are introduced which will come into use afterwards.

Lemma 5. (see [20]) *For an SDD matrix* $\mathbf{C} \in \mathbb{C}^{m\times m}$*, it holds*

$$||\mathbf{C}^{-1}||_\infty \leq \left[\min_{1\leq i\leq m} \mathtt{dd}_i(\mathbf{C})\right]^{-1}.$$

Lemma 6. *Assume there exists two positive numbers l_d and s_d such that $l_d \leq d_{in} \leq s_d$ holds for any $1 \leq i \leq M$ and $1 \leq n \leq N$. Then, for any $\epsilon \in (0, 1]$,*

$$||\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}} - \mathbf{I}_{MN}||_2 < \tau^{-\alpha}\Theta(l_d, s_d)\epsilon,$$

where the positive constant $\Theta(l_d, s_d)$ is given by

$$\Theta(l_d, s_d) = \frac{T^\alpha\Gamma(2-\alpha)}{\sqrt{l_d c_\beta T^\alpha \Gamma(2-\alpha) + (1-\alpha)l_d/s_d}\sqrt{l_d c_\beta T^\alpha\Gamma(2-\alpha) + 1 - \alpha}},$$

and c_β is given by Lemma 3.

Proof: The invertibility of $\mathbf{P}_\epsilon$ is already given by Theorem 4. Recall that

$$\bar{\mathbf{P}} = \mathbf{T}\otimes\mathbf{I}_M + \mathbf{I}_N \otimes (\bar{\mathbf{D}}\mathbf{S}), \quad \bar{\mathbf{D}} = \mathrm{diag}(\bar{d}_1, \bar{d}_2, ..., \bar{d}_M), \quad \bar{d}_i = \frac{1}{N}\sum_{n=1}^{N} d_{in} \geq 0, \quad 1 \leq i \leq M.$$

$$\mathbf{P}_\epsilon = \bar{\mathbf{P}} + \epsilon\widehat{\mathbf{T}}\otimes\mathbf{I}_M = (\mathbf{T} + \epsilon\widehat{\mathbf{T}})\otimes\mathbf{I}_M + \mathbf{I}_N\otimes(\bar{\mathbf{D}}\mathbf{S}).$$

By (3.7) and $\epsilon \in (0, 1]$, it holds

$$\begin{aligned}\min_{1\leq i\leq N} \mathtt{dd}_i(\mathbf{T}+\epsilon\widehat{\mathbf{T}}) &= \min_{1\leq i\leq N}\frac{1}{\tau^\alpha}\left(w_0^{(\alpha)} - \sum_{j=1}^{i-1}|w_j^{(\alpha)}| - \epsilon\sum_{j=i}^{N-1}|w_j^{(\alpha)}|\right)\\ &\geq \frac{(1-\alpha)}{\tau^\alpha\Gamma(2-\alpha)}\int_{N-1}^{N} x^{-\alpha}dx > \frac{(1-\alpha)}{T^\alpha\Gamma(2-\alpha)}.\end{aligned} \tag{3.16}$$

Note that both $\mathbf{T}+\epsilon\widehat{\mathbf{T}}$ and $\bar{\mathbf{D}}\mathbf{S}$ have positive diagonal entries. By Lemma 5, (3.16) and (3.10), it holds

$$||\mathbf{P}_\epsilon^{-1}||_\infty \leq \left[\min_{1\leq i\leq MN}\mathtt{dd}_i(\mathbf{P}_\epsilon)\right]^{-1} \leq \left[\min_{1\leq i\leq MN}\mathtt{dd}_i((\mathbf{T}+\epsilon\widehat{\mathbf{T}})\otimes\mathbf{I}_M) + \min_{1\leq i\leq MN}\mathtt{dd}_i(\mathbf{I}_N\otimes(\bar{\mathbf{D}}\mathbf{S}))\right]^{-1}$$

$$= \left[\min_{1 \le i \le N} \mathtt{dd}_i(\mathbf{T} + \epsilon \widehat{\mathbf{T}}) + \min_{1 \le i \le M} \mathtt{dd}_i(\bar{\mathbf{D}}\mathbf{S}) \right]^{-1}$$
$$< \left[\frac{(1-\alpha)}{T^\alpha \Gamma(2-\alpha)} + l_d c_\beta \right]^{-1}. \tag{3.17}$$

Similar to (3.16), we have

$$\min_{1 \le i \le N} \mathtt{dd}_i(\mathbf{T}^{\mathrm{T}} + \epsilon \widehat{\mathbf{T}}^{\mathrm{T}}) = \min_{1 \le i \le N} \frac{1}{\tau^\alpha} \left(w_0^{(\alpha)} - \epsilon \sum_{j=N-i+1}^{N-1} |w_j^{(\alpha)}| - \sum_{j=1}^{N-i} |w_j^{(\alpha)}| \right)$$
$$\ge \frac{1}{\tau^\alpha} \left(w_0^{(\alpha)} - \sum_{j=1}^{N-1} |w_j^{(\alpha)}| \right) > \frac{(1-\alpha)}{T^\alpha \Gamma(2-\alpha)}. \tag{3.18}$$

Since $\bar{\mathbf{D}}$ is invertible, $\mathbf{P}_\epsilon$ can also be rewritten as

$$\mathbf{P}_\epsilon = (\mathbf{I}_N \otimes \bar{\mathbf{D}})\mathbf{G}, \quad \mathbf{G} = (\mathbf{T} + \epsilon \widehat{\mathbf{T}}) \otimes \bar{\mathbf{D}}^{-1} + \mathbf{I}_N \otimes \mathbf{S}.$$

By (3.18) and (3.10), it is easy to see that $\mathbf{G}$ is also SDD with positive diagonal entries. Again, by Lemma 5, (3.18) and (3.10), it holds

$$||\mathbf{P}_\epsilon^{-\mathrm{T}}||_\infty = ||(\mathbf{I}_N \otimes \bar{\mathbf{D}}^{-1})\mathbf{G}^{-\mathrm{T}}||_\infty$$
$$\le l_d^{-1} ||\mathbf{G}^{-\mathrm{T}}||_\infty$$
$$\le l_d^{-1} \left[\min_{1 \le i \le MN} \mathtt{dd}_i(\mathbf{G}^{\mathrm{T}}) \right]^{-1}$$
$$\le l_d^{-1} \left[\min_{1 \le i \le MN} \mathtt{dd}_i((\mathbf{T}^{\mathrm{T}} + \epsilon \widehat{\mathbf{T}}^{\mathrm{T}}) \otimes \bar{\mathbf{D}}^{-1}) + \min_{1 \le i \le MN} \mathtt{dd}_i(\mathbf{I}_N \otimes \mathbf{S}) \right]^{-1}$$
$$= l_d^{-1} \left[\min_{1 \le i \le N} \mathtt{dd}_i(\mathbf{T}^{\mathrm{T}} + \epsilon \widehat{\mathbf{T}}^{\mathrm{T}}) \min_{1 \le i \le M} \bar{d}_i^{-1} + \min_{1 \le i \le M} \mathtt{dd}_i(\mathbf{S}) \right]^{-1}$$
$$< l_d^{-1} \left[\frac{(1-\alpha)}{s_d T^\alpha \Gamma(2-\alpha)} + c_\beta \right]^{-1}. \tag{3.19}$$

By (3.17) and (3.19), it yields that

$$||\mathbf{P}_\epsilon^{-1}||_2 = \sqrt{\lambda_{\max}(\mathbf{P}_\epsilon^{-\mathrm{T}} \mathbf{P}_\epsilon^{-1})} \le \sqrt{||\mathbf{P}_\epsilon^{-\mathrm{T}} \mathbf{P}_\epsilon^{-1}||_\infty}$$
$$\le \sqrt{||\mathbf{P}_\epsilon^{-\mathrm{T}}||_\infty ||\mathbf{P}_\epsilon^{-1}||_\infty}$$
$$< \left[l_d \left(\frac{(1-\alpha)}{s_d T^\alpha \Gamma(2-\alpha)} + c_\beta \right) \left(\frac{(1-\alpha)}{T^\alpha \Gamma(2-\alpha)} + l_d c_\beta \right) \right]^{-\frac{1}{2}}. \tag{3.20}$$

By (ii) in Lemma 1,

$$||\widehat{\mathbf{T}} \otimes \mathbf{I}_M||_2 = ||\widehat{\mathbf{T}}||_2 = \sqrt{\lambda_{\max}(\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{T}})} \leq \sqrt{||\widehat{\mathbf{T}}^{\mathrm{T}}\widehat{\mathbf{T}}||_\infty} \leq \sqrt{||\widehat{\mathbf{T}}^{\mathrm{T}}||_\infty||\widehat{\mathbf{T}}||_\infty} \leq \frac{1}{\tau^\alpha}\sum_{k=1}^{N-1}|w_k^{(\alpha)}| < \tau^{-\alpha}w_0^{(\alpha)}. \quad (3.21)$$

By (3.20) and (3.21),

$$\begin{aligned}||\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}} - \mathbf{I}_{MN}||_2 &= ||\mathbf{P}_\epsilon^{-1}(\mathbf{P}_\epsilon - \epsilon\widehat{\mathbf{T}} \otimes \mathbf{I}_M) - \mathbf{I}_{MN}||_2 \\ &= ||\epsilon\mathbf{P}_\epsilon^{-1}(\widehat{\mathbf{T}} \otimes \mathbf{I}_M)||_2 \leq \epsilon||\mathbf{P}_\epsilon^{-1}||_2||\widehat{\mathbf{T}} \otimes \mathbf{I}_M||_2 < \tau^{-\alpha}\Theta(l_d, s_d)\epsilon,\end{aligned}$$

which completes the proof. □

For $\mathbf{C} \in \mathbb{C}^{m\times m}$, let $\Sigma(\mathbf{C})$ denote the set of singular values of $\mathbf{C}$ and also denote $\Sigma^2(\mathbf{C}) = \{x^2|x \in \Sigma(\mathbf{C})\}$.

For any Hermitian matrices $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{C}^{m\times m}$, denote $\mathbf{H}_1 \prec$ (or $\preceq$)$\mathbf{H}_2$ if $\mathbf{H}_2-\mathbf{H}_1$ is Hermitian positive definite (or Hermitian positive semi-definite). Especially, we denote $\mathbf{O} \prec$ (or $\preceq$)$\mathbf{H}_1$, when $\mathbf{H}_1$ itself is Hermitian positive definite (or Hermitian positive semi-definite). Similarly, $\mathbf{H}_1 \prec$ (or $\preceq$) $\mathbf{H}_2$ and $\mathbf{O} \prec$ (or $\preceq$)$\mathbf{H}_2$ have the same meanings.

Theorem 7. *Let $d(x,t) \equiv d(x)$ for any $(x,t) \in (a,b) \times (0,T]$. Assume $l_d := \inf_{x\in(a,b)} d(x) > 0$. Denote*

$$\omega = \frac{T^\alpha\Gamma(2-\alpha)}{\sqrt{l_d c_\beta T^\alpha\Gamma(2-\alpha)}\sqrt{l_d c_\beta T^\alpha\Gamma(2-\alpha)+1-\alpha}},$$

with c_β given by Lemma 3. For any $\delta \in (0, \frac{1}{2})$, take $\epsilon \in (0,\vartheta]$ with $\vartheta = \min\left\{\omega^{-1}\delta\tau^\alpha, 1\right\}$. Then, it holds

$$\Sigma^2(\mathbf{P}_\epsilon^{-1}\mathbf{A}) \subset (1-2\delta, (1+\delta)^2).$$

Proof: Since $d(x,t) \equiv d(x)$, it is easy to see that $\mathbf{A}$ is exactly $\bar{\mathbf{P}}$. Denote

$$\mathbf{R} = \mathbf{P}_\epsilon^{-1}\mathbf{A} - \mathbf{I}_{MN}.$$

Since $\epsilon \in (0,\vartheta] \subset (0,1]$, Lemma 6 gives

$$||\mathbf{R}||_2 < \Theta\Big(l_d, \max_{1\leq n\leq N}\max_{1\leq i\leq M} d_{in}\Big)\epsilon\tau^{-\alpha} < \omega\epsilon\tau^{-\alpha} \leq \omega\vartheta\tau^{-\alpha} \leq \delta, \quad (3.22)$$

where $\Theta(\cdot,\cdot)$ is defined in Lemma 6. Thus, $(\mathbf{P}_\epsilon^{-1}\mathbf{A})(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}} = \mathbf{I}_{MN} + \mathbf{R} + \mathbf{R}^{\mathrm{T}} + \mathbf{R}\mathbf{R}^{\mathrm{T}} \succeq \mathbf{I}_{MN} + \mathbf{R} + \mathbf{R}^{\mathrm{T}} \succeq \mathbf{I}_{MN} - 2||\mathbf{R}||_2\mathbf{I}_{MN} \succeq (1-2\delta)\mathbf{I}_{MN}$, which implies that

$$\lambda_{\min}((\mathbf{P}_\epsilon^{-1}\mathbf{A})(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}}) > (1-2\delta). \quad (3.23)$$

By (3.22) again, $||(\mathbf{P}_\epsilon^{-1}\mathbf{A})(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}}||_2 \le 1 + 2||\mathbf{R}||_2 + ||\mathbf{R}||_2^2 < 1 + 2\delta + \delta^2 = (1+\delta)^2$, which implies that

$$\lambda_{\max}((\mathbf{P}_\epsilon^{-1}\mathbf{A})(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}}) < (1+\delta)^2. \tag{3.24}$$

The result follows from (3.23) and (3.24). □

Remark: It is easy to see that ϑ given in Theorem 7 is actually of $\mathcal{O}(\tau^\alpha)$. Hence, Theorem 7 shows that the preconditioned matrix $\mathbf{P}_\epsilon^{-1}\mathbf{A}$ has singular values clustered near 1 without outliers when ϵ is taken to be of $\mathcal{O}(\tau^\alpha)$ and the coefficient function is temporally independent and positive. When the conjugate gradient method is applied to solving the normalized preconditioned linear system with the coefficient matrix $(\mathbf{P}_\epsilon^{-1}\mathbf{A})^T(\mathbf{P}_\epsilon^{-1}\mathbf{A})$, the method converges superlinearly; see [14].

Next we study the uniform boundedness of singular values of the matrix $\mathbf{P}_\epsilon^{-1}\mathbf{A}$ for temporal dependent coefficients. Let $C^k[x,y]$ denote the set of functions with continuous kth order derivative over the interval $[x,y]$.

Lemma 8. (see [24]) *Let $q(t) \in C^2[0,T]$ and $q_n = q(t_n)(0 \le n \le N)$. Then, it holds*

$$\max_{1\le n\le N} \left|{}_0^C D_{t_n}^\alpha q(t) - D_\tau^\alpha q_n\right| \le B_{\alpha,q}\tau^{2-\alpha},$$

where the discrete operator D_τ^α is defined in (2.1) *and*

$$B_{\alpha,q} = \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - 1 - 2^{-\alpha}\right]\max_{t\in[0,T]}\left|q''(t)\right|.$$

Lemma 9. *Let $q(t)$ be a function defined on $[0,T]$. Denote $\mathbf{Q} = \mathrm{diag}(q_1, q_2, ..., q_N)$ with $q_n = q(t_n)$ $(0 \le n \le N)$. Assume that*

(i) $q \in C^2[0,T]$,

(ii) $\min_{t\in[0,T]} q(t) \ge 0$ *and* $q(T) > 0$,

(iii) $\frac{c_\alpha q(0)}{T^\alpha} + \inf_{t\in(0,T)}\left[\frac{c_\alpha q(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha q(t)\right] > 0$ *with c_α given by Lemma 1.*

Then, there exists a positive constant N_0 depending only on α and q such that for any $N \ge N_0$, it holds

$$\mathbf{TQ} + \mathbf{QT}^{\mathrm{T}} \quad \text{is Hermitian positive semi-definite.}$$

Proof: Denote $\mathbf{H} = \mathbf{TQ} + \mathbf{QT}^{\mathrm{T}} = [h_{ij}]_{i,j=1}^N$. By (i) in Lemma 1 and (ii),

$$h_{ij} = \begin{cases} q_j\tau^{-\alpha}w_{i-j}^{(\alpha)} \le 0, & i > j, \\ 2q_i\tau^{-\alpha}w_0^{(\alpha)} \ge 0, & i = j, \\ q_i\tau^{-\alpha}w_{j-i}^{(\alpha)} \le 0, & i < j. \end{cases} \tag{3.25}$$

For any $\mathbf{z} = (z_1, z_2, ..., z_N)^{\mathrm{T}} \in \mathbb{C}^{N\times 1}$, by (3.25), it holds that

$$
\begin{aligned}
\mathbf{z}^*\mathbf{H}\mathbf{z} = \sum_{i,j=1}^{N} \bar{z}_i h_{ij} z_j &= \sum_{i=1}^{N} h_{ii}|z_i|^2 + \sum_{i=2}^{N}\sum_{j=1}^{i-1} \bar{z}_i h_{ij} z_j + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \bar{z}_i h_{ij} z_j \\
&= \sum_{i=1}^{N} h_{ii}|z_i|^2 + 2\sum_{i=2}^{N}\sum_{j=1}^{i-1} \mathrm{Re}\left(\bar{z}_i h_{ij} z_j\right) \\
&\geq \sum_{i=1}^{N} h_{ii}|z_i|^2 - \sum_{i=2}^{N}\sum_{j=1}^{i-1} |h_{ij}|(|z_i|^2 + |z_j|^2) \\
&= |z_1|^2\left(h_{11} - \sum_{j=2}^{N}|h_{j1}|\right) + \sum_{i=2}^{N-1}|z_i|^2\left(h_{ii} - \sum_{j=1}^{i-1}|h_{ij}| - \sum_{j=i+1}^{N}|h_{ji}|\right) + \\
&\quad |z_N|^2\left(h_{NN} - \sum_{j=1}^{N-1}|h_{Nj}|\right) \\
&= \frac{|z_1|^2 q_1}{\tau^\alpha}\left(w_0^{(\alpha)} + \sum_{j=0}^{N-1} w_j^{(\alpha)}\right) + \frac{|z_N|^2}{\tau^\alpha}\left(q_N w_0^{(\alpha)} + \sum_{j=1}^{N} q_j w_{N-j}^{(\alpha)}\right) + \\
&\quad \sum_{i=2}^{N-1}\frac{|z_i|^2}{\tau^\alpha}\left(q_i\sum_{j=0}^{N-i} w_j^{(\alpha)} + \sum_{j=1}^{i} w_{i-j}^{(\alpha)} q_j\right).
\end{aligned}
\tag{3.26}
$$

By (ii) and Lemma 1,

$$
\frac{q_1}{\tau^\alpha}\left(w_0^{(\alpha)} + \sum_{j=0}^{N-1} w_j^{(\alpha)}\right) \geq 0. \tag{3.27}
$$

By definition of D_τ^α, $q(0) \geq 0$ and Lemma 8,

$$
\begin{aligned}
\frac{1}{\tau^\alpha}\left(q_N w_0^{(\alpha)} + \sum_{j=1}^{N} q_j w_{N-j}^{(\alpha)}\right) &= \frac{q(T)w_0^{(\alpha)}}{\tau^\alpha} + D_\tau^\alpha q_N + \frac{[N^{1-\alpha} - (N-1)^{1-\alpha}]q(0)}{\tau^\alpha\Gamma(2-\alpha)} \\
&\geq \tau^{-\alpha} q(T) w_0^{(\alpha)} + {}_0^C D_T^\alpha q(t) + D_\tau^\alpha q_N - {}_0^C D_T^\alpha q(t) \\
&\geq \tau^{-\alpha} q(T) w_0^{(\alpha)} + {}_0^C D_T^\alpha q(t) - B_{\alpha,q}\tau^{2-\alpha}.
\end{aligned}
\tag{3.28}
$$

By (i),

$$
\begin{aligned}
\left|{}_0^C D_T^\alpha q(t)\right| \leq \sup_{t\in(0,T]}\left|{}_0^C D_t^\alpha q(t)\right| &= [\Gamma(1-\alpha)]^{-1}\sup_{t\in(0,T]}\left|\int_0^t |q'(s)|(t-s)^{-\alpha}ds\right| \\
&\leq [\Gamma(1-\alpha)]^{-1}\max_{t\in[0,T]}|q'(t)|\sup_{t\in(0,T]}\left|\int_0^t (t-s)^{-\alpha}ds\right|
\end{aligned}
$$

$$= [\Gamma(2-\alpha)]^{-1} \max_{t\in[0,T]} |q'(t)| T^{1-\alpha} < +\infty. \tag{3.29}$$

By $q(T) > 0$ and $q \in C^2[0,T]$, it holds

$$\lim_{N\to+\infty} \left[q(T) w_0^{(\alpha)} \tau^{-\alpha} - B_{\alpha,q}\tau^{2-\alpha} \right] = +\infty. \tag{3.30}$$

From (3.28)–(3.30), it is easy to see that there exists a positive constant N_1 depending on α and q such that for any $N \geq N_1$, it holds

$$\frac{1}{\tau^\alpha} \left(q_N w_0^{(\alpha)} + \sum_{j=1}^{N} q_j w_{N-j}^{(\alpha)} \right) \geq 0. \tag{3.31}$$

Denote

$$p_\alpha(m) = \left(1 + \frac{1}{2m}\right)^\alpha, \quad m > 0, \qquad L_q(i) = {}_0^C D_{t_i}^\alpha q(t) - B_{\alpha,q}\tau^{2-\alpha} + \frac{c_\alpha q(0)}{T^\alpha}, \quad 2 \leq i \leq N-1.$$

Again, by $q(0) \geq 0$, (ii) in Lemma 1 and Lemma 8, it holds

$$\begin{aligned}
\frac{1}{\tau^\alpha}\left(q_i \sum_{j=0}^{N-i} w_j^{(\alpha)} + \sum_{j=1}^{i} w_{i-j}^{(\alpha)} q_j \right) &= \frac{q_i}{\tau^\alpha} \sum_{j=0}^{N-i} w_j^{(\alpha)} + D_\tau^\alpha q_i + \frac{(1-\alpha) q(0)}{\tau^\alpha \Gamma(2-\alpha)} \int_{i-1}^{i} x^{-\alpha} dx \\
&\geq \frac{c_\alpha q_i}{\tau^\alpha (N - i + \frac{1}{2})^\alpha} + D_\tau^\alpha q_i + \frac{c_\alpha q(0)}{T^\alpha} \\
&= \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + {}_0^C D_{t_i}^\alpha q(t) + D_\tau^\alpha q_i - {}_0^C D_{t_i}^\alpha q(t) + \frac{c_\alpha q(0)}{T^\alpha} \\
&\geq \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + L_q(i), \quad 2 \leq i \leq N-1.
\end{aligned} \tag{3.32}$$

(3.29) and $q \in C^2[0,T]$ imply that

$$\sup_{N\geq 3} \max_{2\leq i\leq N-1} |L_q(i)| \leq \sup_{t\in(0,T]} \left| {}_0^C D_t^\alpha q(t) \right| + \sup_{N\geq 1} \left| B_{\alpha,q}\tau^{2-\alpha} \right| + \frac{c_\alpha q(0)}{T^\alpha} < +\infty. \tag{3.33}$$

On the other hand, $q(T) > 0$ and continuity of $q(t)$ induce that $\lim_{t\to T} \frac{2^\alpha c_\alpha q(t)}{3^\alpha (T-t)^\alpha} = \infty$. Thus, there exists a constant $\eta \in (0,T)$ such that for any $t \in [\eta, T)$, it holds

$$\frac{2^\alpha c_\alpha q(t)}{3^\alpha (T-t)^\alpha} \geq \sup_{t\in(0,T]} \left| {}_0^C D_t^\alpha q(t) \right| + \sup_{N\geq 1} \left| B_{\alpha,q}\tau^{2-\alpha} \right| + \frac{c_\alpha q(0)}{T^\alpha} + 1. \tag{3.34}$$

Note that $\{t_2, t_3, ..., t_{N-1}\} \subset (0,T) = (0,\eta) \cup [\eta, T)$. For $t_i \in [\eta, T)$ and $2 \le i \le N-1$, by (3.34) and (3.33), it holds

$$\frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + L_q(i) \ge \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(1)} + L_q(i) = \frac{2^\alpha c_\alpha q(t)}{3^\alpha (T-t_i)^\alpha} + L_q(i) \ge 1. \tag{3.35}$$

Denote $c_0 = \max_{t\in[0,T]} |q(t)|$. For $t_i \in (0,\eta)$, it holds $i < \eta\tau^{-1}$ and thus

$$\begin{aligned}
\frac{c_\alpha q(t_i)\left[p_\alpha^{-1}(N-i)-1\right]}{(T-t_i)^\alpha} \ge \frac{c_\alpha c_0\left[p_\alpha^{-1}(N-i)-1\right]}{(T-\eta)^\alpha} &> \frac{c_\alpha c_0}{(T-\eta)^\alpha}\left[p_\alpha^{-1}\left(T\tau^{-1}-\eta\tau^{-1}\right)-1\right] \\
&\ge \frac{c_\alpha c_0}{(T-\eta)^\alpha}\left\{\left[1+\frac{\tau}{2(T-\eta)}\right]^{-1}-1\right\} \\
&= \frac{-c_\alpha c_0 \tau}{(T-\eta)^\alpha\left[2(T-\eta)+\tau\right]} \ge \frac{-c_\alpha c_0\tau}{2(T-\eta)^{1+\alpha}}.
\end{aligned}$$

Thus, for any $t_i \in (0,\eta)$, it holds that

$$\begin{aligned}
\frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + L_q(i) &= L_q(i) + \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha} + \frac{c_\alpha q(t_i)\left[p_\alpha^{-1}(N-i)-1\right]}{(T-t_i)^\alpha} \\
&\ge L_q(i) + \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha} - \frac{c_\alpha c_0\tau}{2(T-\eta)^{1+\alpha}} \\
&\ge \frac{c_\alpha q(0)}{T^\alpha} + \inf_{t\in(0,T)}\left[\frac{c_\alpha q(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha q(t)\right] - \frac{c_\alpha c_0\tau}{2(T-\eta)^{1+\alpha}} - B_{\alpha,q}\tau^{2-\alpha}.
\end{aligned}$$

By (iii), $\frac{c_\alpha q(0)}{T^\alpha} + \inf_{t\in(0,T)}\left[\frac{c_\alpha q(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha q(t)\right] > 0$. Moreover, $\lim_{N\to+\infty}\left|-\frac{c_\alpha c_0\tau}{2(T-\eta)^{1+\alpha}} - B_{\alpha,q}\tau^{2-\alpha}\right| = 0$. Hence, there exists a positive constant N_2 depending only on α and q such that for any $N \ge N_2$ and any $t_i \in (0,\eta)$

$$\frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + L_q(i) > 0,$$

which together with (3.35) and (3.28) implies that for any $N \ge N_2$

$$\frac{1}{\tau^\alpha}\left(q_i\sum_{j=0}^{N-i} w_j^{(\alpha)} + \sum_{j=1}^{i} w_{i-j}^{(\alpha)} q_j\right) \ge \frac{c_\alpha q(t_i)}{(T-t_i)^\alpha p_\alpha(N-i)} + L_q(i) > 0, \quad 2 \le i \le N-1. \tag{3.36}$$

By (3.26), (3.27), (3.31) and (3.36), taking $N_0 = \max\{N_1, N_2\}$ yields that for any $N \ge N_0$

$$\mathbf{z}^*\mathbf{H}\mathbf{z} \ge 0,$$

which completes the proof. □

Lemma 10. *Let $q(t)$ be a function defined on $[0,T]$. Assume that*

(i) $q \in C^2[0,T]$,

(ii) $l_q := \min\limits_{t\in[0,T]} q(t) > 0$,

(iii) $\nu := \frac{c_\alpha q(0)}{T^\alpha} + \inf\limits_{t\in(0,T)} \left[\frac{c_\alpha q(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha q(t)\right] > 0$ *with* c_α *given by Lemma 1.*

Denote $\mathbf{Q} = \mathrm{diag}(q_1, q_2, ..., q_N)$ *with* $q_n = q(t_n)(0 \le n \le N)$. *Then, there exists a positive constant* N_0 *independent of* τ *and* h *such that for any* $N \ge N_0$

$$\mathbf{O} \prec \theta\nu(\mathbf{T} + \mathbf{T}^{\mathrm{T}}) \preceq \mathbf{T}\mathbf{Q} + \mathbf{Q}\mathbf{T}^{\mathrm{T}} \preceq \kappa_q(\mathbf{T} + \mathbf{T}^{\mathrm{T}}),$$

where θ, κ_q *defined as follows are positive constants independent of* τ *and* h,

$$\theta = \min\left\{\frac{T^\alpha}{4c_\alpha}, \frac{(T-\eta)^\alpha}{4c_\alpha}, \frac{l_q}{2\nu}\right\}, \quad \eta = \max\left\{\frac{T}{2}, T - \left[\frac{c_\alpha l_q}{2(\nu + c_q)}\right]^{\frac{1}{\alpha}}\right\},$$
$$\kappa_q = c_\alpha^{-1} c_q T^\alpha + 2s_q, \quad c_q = [\Gamma(2-\alpha)]^{-1} T^{1-\alpha} \max_{t\in[0,T]} |q'(t)|, \quad s_q = \max_{t\in(0,T)} q(t).$$

Proof: For any smooth function $w(t)$ defined on $[0,T]$, define the linear transformation

$$G_w(t) = \frac{c_\alpha w(0)}{T^\alpha} + \frac{c_\alpha w(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha w(t).$$

Similar to (3.29), it holds

$$\sup_{t\in(0,T]} \left|{}_0^C D_t^\alpha q(t)\right| \le c_q. \tag{3.37}$$

Hence,

$$\begin{aligned}
\inf_{t\in(0,T)} G_{\kappa_q - q}(t) > \inf_{t\in(0,T)} \left\{\frac{c_\alpha[\kappa_q - q(t)]}{(T-t)^\alpha} - {}_0^C D_t^\alpha q(t)\right\} &> \inf_{t\in(0,T)} \left\{\frac{c_\alpha[\kappa_q - s_q]}{(T-t)^\alpha} - c_q\right\} \\
&= \inf_{t\in(0,T)} \left[\frac{c_\alpha s_q}{(T-t)^\alpha} + \frac{c_q T^\alpha}{(T-t)^\alpha} - c_q\right] \\
&\ge \inf_{t\in(0,T)} \frac{c_\alpha s_q}{(T-t)^\alpha} > 0.
\end{aligned}$$

Note also that $\min\limits_{t\in[0,T]}[\kappa_q - q(t)] \ge \min\limits_{t\in[0,T]}[2s_q - q(t)] > 0$ and $\kappa_q - q(t) \in C^2[0,T]$. Thus, by Lemma 9, there exists a positive constant N_1 depending on q and α such that for any $N \ge N_1$

$$\mathbf{O} \preceq (\kappa_q \mathbf{I}_N - \mathbf{Q})\mathbf{T}^{\mathrm{T}} + \mathbf{T}(\kappa_q \mathbf{I}_N - \mathbf{Q}). \tag{3.38}$$

On the other hand,

$$\inf_{t\in(\eta,T)} G_{q-\theta\nu}(t) = \inf_{t\in(\eta,T)} \left[\frac{c_\alpha(q(0) - \theta\nu)}{T^\alpha} + \frac{c_\alpha(q(t) - \theta\nu)}{(T-t)^\alpha} + {}_0^C D_t^\alpha q(t)\right]$$

$$
\begin{aligned}
&\geq \frac{c_\alpha l_q}{2(T-\eta)^\alpha} - c_q \\
&\geq \frac{c_\alpha l_q}{2\left(T - T + \left[\frac{c_\alpha l_q}{2(\nu+c_q)}\right]^{\frac{1}{\alpha}}\right)^\alpha} - c_q = \nu,
\end{aligned} \tag{3.39}
$$

where the first inequality is induced by $\theta\nu \leq 2^{-1} l_q$ and (3.37), the second inequality is obtained from $\eta \geq T - \left[\frac{c_\alpha l_q}{2(\nu+c_q)}\right]^{\frac{1}{\alpha}}$. Moreover, for any $t \in (0, \eta]$, it holds

$$
\nu = \frac{c_\alpha}{T^\alpha} \times \frac{T^\alpha \nu}{2c_\alpha} + \frac{c_\alpha}{(T-t)^\alpha} \times \frac{(T-t)^\alpha \nu}{2c_\alpha} > \frac{c_\alpha \theta\nu}{T^\alpha} + \frac{c_\alpha \theta\nu}{(T-t)^\alpha} = G_{\theta\nu}(t).
$$

Above formula and (iii) induce

$$
\inf_{t\in(0,\eta]} G_{q-\theta\nu}(t) = \inf_{t\in(0,\eta]} [G_q(t) - G_{\theta\nu}(t)] \geq \inf_{t\in(0,T)} G_q(t) - \sup_{t\in(0,\eta]} G_{\theta\nu}(t) = \nu - \sup_{t\in(0,\eta]} G_{\theta\nu}(t) > 0,
$$

which together with (3.39) implies that $\inf\limits_{t\in(0,T)} G_{q-\theta\nu}(t) > 0$. Note also that $q - \theta\nu \in C^2[0,T]$ and $\min\limits_{t\in[0,T]} [q(t) - \theta\nu] \geq 2^{-1} l_q > 0$. By Lemma 9 again, there exists a positive constant N_2 depending on α and q such that for any $N \geq N_2$,

$$
\mathbf{O} \preceq (\mathbf{Q} - \theta\nu \mathbf{I}_N)\mathbf{T}^{\mathrm{T}} + \mathbf{T}(\mathbf{Q} - \theta\nu \mathbf{I}_N). \tag{3.40}
$$

By (3.38) and (3.40), taking $N_0 = \max\{N_1, N_2\}$ yields that for any $N \geq N_0$,

$$
\theta\nu(\mathbf{T} + \mathbf{T}^{\mathrm{T}}) \preceq \mathbf{T}\mathbf{Q} + \mathbf{Q}\mathbf{T}^{\mathrm{T}} \preceq \kappa_q(\mathbf{T} + \mathbf{T}^{\mathrm{T}}).
$$

From Lemma 1, it is easy to see that $\mathbf{T} + \mathbf{T}^{\mathrm{T}}$ is SDD with positive diagonal entries. By Gershgorin circle theorem, $\mathbf{T} + \mathbf{T}^{\mathrm{T}}$ is Hermitian positive definite and thus $\theta\nu(\mathbf{T} + \mathbf{T}^{\mathrm{T}}) \succ \mathbf{O}$, which completes the proof. □

Proposition 1. *For positive numbers θ_i, η_i ($1 \leq i \leq m$), it obviously holds that*

$$
\min_{1\leq i\leq m} \frac{\theta_i}{\eta_i} \leq \left(\sum_{i=1}^m \eta_i\right)^{-1} \left(\sum_{i=1}^m \theta_i\right) \leq \max_{1\leq i\leq m} \frac{\theta_i}{\eta_i}.
$$

Theorem 11. *Let $d(x,t) \equiv d(t)$ for any $(x,t) \in (a,b) \times (0,T]$. Assume that*

(i) $d(t) \in C^2[0,T]$,

(ii) $l_d := \min\limits_{t\in[0,T]} d(t) > 0$,

(iii) $\nu := \frac{c_\alpha d(0)}{T^\alpha} + \inf\limits_{t\in(0,T)} \left[\frac{c_\alpha d(t)}{(T-t)^\alpha} + {}_0^C D_t^\alpha d(t)\right] > 0$, *with c_α given by Lemma 1.*

Then there exists a positive constant N_0 independent of τ and h such that for any $\epsilon \in (0, \mu]$ and any $N \geq N_0$,

$$\Sigma^2(\mathbf{P}_\epsilon^{-1}\mathbf{A}) \subset (c_1, c_2),$$

where c_1 and c_2 defined as follows are positive constants independent of τ and h,

$$\mu = \min\left\{\frac{T^\alpha\Gamma(2-\alpha)\tau^\alpha}{4l_d c_\beta T^\alpha\Gamma(2-\alpha)+1-\alpha}, 1\right\}, \quad c_1 = \frac{16}{25}\min\left\{\frac{\theta\nu}{s_d}, \frac{l_d^2}{s_d^2}\right\} > 0,$$

$$\theta = \min\left\{\frac{T^\alpha}{4c_\alpha}, \frac{(T-\eta)^\alpha}{4c_\alpha}, \frac{l_d}{2\nu}\right\}, \quad \eta = \max\left\{\frac{T}{2}, T - \left[\frac{c_\alpha l_d}{2(\nu+c_d)}\right]^{\frac{1}{\alpha}}\right\}, \quad c_2 = 2\max\left\{\frac{\kappa_d}{l_d}, \frac{s_d^2}{l_d^2}\right\},$$

$$\kappa_d = c_\alpha^{-1} c_d T^\alpha + 2s_d, \quad c_d = [\Gamma(2-\alpha)]^{-1}T^{1-\alpha}\max_{t\in[0,T]}|d'(t)|, \quad s_d = \max_{t\in[0,T]} d(t),$$

c_β is given in Lemma 3.

Proof: Let $\mathbf{D}_t = \mathrm{diag}(d_1, d_2, ..., d_N)$ with $d_n = d(t_n)(0 \leq n \leq N)$. Then,

$$\mathbf{A} = \mathbf{T}\otimes\mathbf{I}_M + \mathbf{D}_t\otimes\mathbf{S}, \quad \bar{\mathbf{P}} = \mathbf{T}\otimes\mathbf{I}_M + \bar{d}\mathbf{I}_N\otimes\mathbf{S}, \quad \bar{d} = \frac{1}{N}\sum_{n=1}^{N} dn,$$

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = (\mathbf{T}\mathbf{T}^{\mathrm{T}})\otimes\mathbf{I}_M + (\mathbf{D}_t\mathbf{T}^{\mathrm{T}} + \mathbf{T}\mathbf{D}_t)\otimes\mathbf{S} + \mathbf{D}_t^2\otimes\mathbf{S}^2,$$

$$\bar{\mathbf{P}}\bar{\mathbf{P}}^{\mathrm{T}} = (\mathbf{T}\mathbf{T}^{\mathrm{T}})\otimes\mathbf{I}_M + \bar{d}(\mathbf{T}^{\mathrm{T}} + \mathbf{T})\otimes\mathbf{S} + \bar{d}^2\mathbf{I}_N\otimes\mathbf{S}^2.$$

By Lemma 6, for any $\epsilon \in (0, \mu]$, it holds

$$||\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}} - \mathbf{I}_{MN}||_2 < \Theta(\bar{d}, \bar{d})\epsilon\tau^{-\alpha} \leq \Theta(l_d, l_d)\epsilon\tau^{-\alpha} \leq \Theta(l_d, l_d)\mu\tau^{-\alpha} \leq \frac{1}{4},$$

where $\Theta(\cdot,\cdot)$ is defined in Lemma 6. Thus, similar to the proof of Theorem 7, one can prove that

$$\Sigma^2(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}}) \subset \left(\frac{1}{2}, \frac{16}{25}\right). \tag{3.41}$$

By Lemma 10, there exists a positive constant N_0 depending only on α and d such that for any $N \geq N_0$,

$$\mathbf{O} \prec \theta\nu(\mathbf{T} + \mathbf{T}^{\mathrm{T}}) \preceq \mathbf{D}_t\mathbf{T}^{\mathrm{T}} + \mathbf{T}\mathbf{D}_t \preceq \kappa_d(\mathbf{T}^{\mathrm{T}} + \mathbf{T}), \tag{3.42}$$

It is easy to see that $\bar{\mathbf{P}}$ is SDD. Thus, $\bar{\mathbf{P}}$ is invertible. For any non-zero vector $\mathbf{x} \in \mathbb{C}^{MN\times 1}$, let $\mathbf{y} = \bar{\mathbf{P}}^{-\mathrm{T}}\mathbf{x}$. By (3.42) and Proposition 1, it holds for any $N \geq N_0$ that

$$\frac{\mathbf{x}^*\bar{\mathbf{P}}^{-1}\mathbf{A}(\bar{\mathbf{P}}^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \frac{\mathbf{y}^*\mathbf{A}\mathbf{A}^{\mathrm{T}}\mathbf{y}}{\mathbf{y}^*\bar{\mathbf{P}}\bar{\mathbf{P}}^{\mathrm{T}}\mathbf{y}}$$

$$
\begin{aligned}
&= \frac{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \mathbf{y}^*[(\mathbf{D}_t\mathbf{T}^{\mathrm{T}} + \mathbf{T}\mathbf{D}_t) \otimes \mathbf{S}]\mathbf{y} + \mathbf{y}^*[\mathbf{D}_t^2 \otimes \mathbf{S}^2]\mathbf{y}}{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \bar{d}\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + \bar{d}^2\mathbf{y}^*(\mathbf{I}_N \otimes \mathbf{S})\mathbf{y}} \\
&\leq \frac{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \kappa_d\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + s_d^2\mathbf{y}^*[\mathbf{I}_N \otimes \mathbf{S}^2]\mathbf{y}}{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + l_d\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + l_d^2\mathbf{y}^*(\mathbf{I}_N \otimes \mathbf{S})\mathbf{y}} \\
&\leq \max\left\{1, \frac{\kappa_d}{l_d}, \frac{s_d^2}{l_d^2}\right\} = \max\left\{\frac{\kappa_d}{l_d}, \frac{s_d^2}{l_d^2}\right\}, \qquad (3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathbf{x}^*\bar{\mathbf{P}}^{-1}\mathbf{A}(\bar{\mathbf{P}}^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{x}}{\mathbf{x}^*\mathbf{x}} &= \frac{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \mathbf{y}^*[(\mathbf{D}_t\mathbf{T}^{\mathrm{T}} + \mathbf{T}\mathbf{D}_t) \otimes \mathbf{S}]\mathbf{y} + \mathbf{y}^*[\mathbf{D}_t^2 \otimes \mathbf{S}^2]\mathbf{y}}{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \bar{d}\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + \bar{d}^2\mathbf{y}^*(\mathbf{I}_N \otimes \mathbf{S})\mathbf{y}} \\
&\geq \frac{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + \theta\nu\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + l_d^2\mathbf{y}^*[\mathbf{I}_N \otimes \mathbf{S}^2]\mathbf{y}}{\mathbf{y}^*[(\mathbf{T}\mathbf{T}^{\mathrm{T}}) \otimes \mathbf{I}_M]\mathbf{y} + s_d\mathbf{y}^*[(\mathbf{T}^{\mathrm{T}} + \mathbf{T}) \otimes \mathbf{S}]\mathbf{y} + s_d^2\mathbf{y}^*(\mathbf{I}_N \otimes \mathbf{S})\mathbf{y}} \\
&\geq \min\left\{1, \frac{\theta\nu}{s_d}, \frac{l_d^2}{s_d^2}\right\} = \min\left\{\frac{\theta\nu}{s_d}, \frac{l_d^2}{s_d^2}\right\}. \qquad (3.44)
\end{aligned}
$$

For any non-zero vector $\mathbf{w} \in \mathbb{C}^{MN\times 1}$, denote $\mathbf{z} = (\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})^{\mathrm{T}}\mathbf{w}$. Then,

$$
\frac{\mathbf{w}^*\mathbf{P}_\epsilon^{-1}\mathbf{A}(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{w}}{\mathbf{w}^*\mathbf{w}} = \frac{\mathbf{w}^*(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})(\bar{\mathbf{P}}^{-1}\mathbf{A})[(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})(\bar{\mathbf{P}}^{-1}\mathbf{A})]^{\mathrm{T}}\mathbf{w}}{\mathbf{z}^*[(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})^{\mathrm{T}}(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})]^{-1}\mathbf{z}} = \frac{\mathbf{z}^*(\bar{\mathbf{P}}^{-1}\mathbf{A})(\bar{\mathbf{P}}^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{z}}{\mathbf{z}^*[(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})^{\mathrm{T}}(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})]^{-1}\mathbf{z}}.
$$

Furthermore, by (3.41), (3.43) and (3.44), it holds for any $N \geq N_0$ that

$$
\begin{aligned}
c_1 = \left(\min\left\{\frac{\theta\nu}{s_d}, \frac{l_d^2}{s_d^2}\right\}||\mathbf{z}||_2^2\right)\Big/\left(\frac{16||\mathbf{z}||_2^2}{25}\right) &< \frac{\mathbf{z}^*(\bar{\mathbf{P}}^{-1}\mathbf{A})(\bar{\mathbf{P}}^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{z}}{\mathbf{z}^*[(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})^{\mathrm{T}}(\mathbf{P}_\epsilon^{-1}\bar{\mathbf{P}})]^{-1}\mathbf{z}} \\
&< \left(\max\left\{\frac{\kappa_d}{l_d}, \frac{s_d^2}{l_d^2}\right\}||\mathbf{z}||_2^2\right)\Big/\left(\frac{||\mathbf{z}||_2^2}{2}\right) = c_2.
\end{aligned}
$$

Thus, $c_1 \leq \frac{\mathbf{w}^*\mathbf{P}_\epsilon^{-1}\mathbf{A}(\mathbf{P}_\epsilon^{-1}\mathbf{A})^{\mathrm{T}}\mathbf{w}}{\mathbf{w}^*\mathbf{w}} \leq c_2$ holds for any $N \geq N_0$, which implies that $\Sigma^2(\mathbf{P}_\epsilon^{-1}\mathbf{A}) \subset (c_1, c_2)$ holds whenever $N \geq N_0$. □

Remark: It is easy to see that μ given in Theorem 11 is of $\mathcal{O}(\tau^\alpha)$. Thus, Theorem 11 actually shows that the singular values of the preconditioned matrix are bounded above and below by mesh-parameters-independent positive constants when ϵ is taken to be of $\mathcal{O}(\tau^\alpha)$ and the coefficient function satisfies the required assumptions in Theorem 12. When the conjugate gradient method is employed to solve the normalized preconditioned system, the method will converge within a constant number of iterations which is independent of τ and h.

4. Implementation

In this section, we present a V-cycle multigrid method to solve the inner problem (3.4). We note that (3.4) can be partitioned as subproblems such that

$$\mathbf{\Lambda}_k \mathbf{x} = \mathbf{y}, \tag{4.1}$$

for some given right hand side $\mathbf{y}$ and some $k \in \{0, 1, ..., N-1\}$.

Denote $M_i = 2^i - 1$ for $i \geq 2$. Let $M = M_l$ for some $l \geq 2$. Also, denote by $\mathbf{\Lambda}_k^{(i)}$, $\mathbf{\Lambda}_k$ equipped with $M = M_i$. Let $\mathbf{I}_{i-1}^i$ and $\mathbf{I}_i^{i-1}$ denote interpolation operator and restriction operator between ith and $(i+1)$th grid; see [2]. For the choice of smoothing iteration, $\mathscr{S}_i$ at ith grid level, we introduce a banded smoothing iteration in the following. Denote $\mathbf{\Lambda}_k^{(i)} = [\lambda_{jk}^{(i)}]_{j,k=1}^{M_i}$. Let $\mathbf{E}_i = [e_{jk}^{(i)}]_{j,k=1}^{M_i}$ denote the tridiagonal truncation of $\mathbf{\Lambda}_k^{(i)}$ such that

$$e_{jk}^{(i)} = \begin{cases} \lambda_{jk}^{(i)}, & |j-k| \leq 1, \\ 0, & |j-k| > 1. \end{cases}$$

For a linear system $\mathbf{\Lambda}_k^{(i)} \mathbf{v} = \mathbf{w}$ with arbitrary given right hand side $\mathbf{w} \in \mathbb{C}^{M_i \times 1}$, it has splitting form such that $\mathbf{E}_i \mathbf{v} = (\mathbf{E}_i - \mathbf{\Lambda}_k^{(i)})\mathbf{v} + \mathbf{w}$, which induces a banded smoothing iteration $\mathscr{S}_i$ as follows

$$\mathbf{v}^{k+1} = \mathscr{S}_i(\mathbf{v}^k, \mathbf{w}) := \mathbf{E}_i^{-1}(\mathbf{E}_i - \mathbf{\Lambda}_k^{(i)})\mathbf{v}^k + \mathbf{E}_i^{-1}\mathbf{w} = \mathbf{v}^k + \mathbf{E}_i^{-1}(\mathbf{w} - \mathbf{\Lambda}_k^{(i)}\mathbf{v}^k), \quad i \geq 3, \tag{4.2}$$

with initial guess $\mathbf{v}^k$. By (3.15) and (3.10), $\mathbf{\Lambda}_k^{(i)}$ is SDD and so is $\mathbf{E}_i$. Hence, $\mathbf{E}_i$ is invertible and (4.2) is well-defined. Since $\mathbf{\Lambda}_k^{(i)}$ is Toeplitz-like and $\mathbf{E}_i$ is banded, it is easy to see that (4.2) requires $\mathcal{O}(M_i \log M_i)$ operation and $\mathcal{O}(M_i)$ storage.

With $\mathbf{\Lambda}_k^{(i)}$, $\mathbf{I}_{i-1}^i$, $\mathbf{I}_i^{i-1}$ and $\mathscr{S}_i$ defined above, one can find algorithm of the V-cycle multigrid method in [2]. We only use one iteration of V-cycle multigrid method with one pre-smoothing and one post-smoothing iteration to solve (4.1) and it requires $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage. Since there are N linear systems of form (4.1) to be solved in (3.4), computation of (3.4) requires $\mathcal{O}(MN \log M)$ operation cost and $\mathcal{O}(MN)$ storage. Therefore, we conclude that computation of $\mathbf{P}_\epsilon^{-1}\mathbf{z}$ using a combination of (3.3), the V-cycle multigrid method and (3.5) requires only $\mathcal{O}(MN)$ storage and $\mathcal{O}(NM \log(MN))$ operation cost.

5. Numerical Results

In this section, we use one example to test our proposed preconditioner and compare the proposed method with other solver. The generalized minimal residual method is applied to solve the preconditioned linear system. All numerical experiments are performed via MATLAB R2015a on a Win10-64-bit PC with the configuration: Intel(R) Core(TM) i7-3770 CPU 3.40 GHz and 16 GB RAM.

Define the relative error

$$\mathrm{E}_{N,M} = \frac{||\mathbf{u} - \tilde{\mathbf{u}}||_\infty}{||\mathbf{u}||_\infty},$$

where $\mathbf{u}$ and $\tilde{\mathbf{u}}$ denote the exact solution and iterative solution of linear system (2.12) deriving from some iterative solvers. As suggested in Theorem 7 and Theorem 11, we set $\epsilon = 10^{-1} \times \tau^\alpha$. Set $\mathbf{P}_\epsilon^{-1}\mathbf{f}$ as initial guess of our proposed method. Set $\frac{||\mathbf{r}_k||_2}{||\mathbf{r}_0||_2} \leq 10^{-7}$ as stopping criterion for generalized minimal residual method, where $\mathbf{r}_k$ denotes the residual vector at kth iteration. Denote by 'iter', the iteration number of generalized minimal residual method with the proposed preconditioner. Denote by 'CPU', the running time by unit second. In the following, we use SEP to denote the proposed separable preconditioner and the corresponding solver.

As mentioned in introduction part, the BFS method with PGMRES inner solver proposed in [26] can also be extended to solve the linear system (2.12). In details, their method can be described as follows. Denote $\tilde{w}_k^{(\alpha)} = \tau^{-\alpha} w_k^{(\alpha)}$ for all $k \geq 0$. Partition $\mathbf{f}$ in (2.12) as

$$\mathbf{f} = (\tilde{\mathbf{f}}_1^{\mathrm{T}}, \tilde{\mathbf{f}}_2^{\mathrm{T}}, ..., \tilde{\mathbf{f}}_N^{\mathrm{T}})^{\mathrm{T}}, \quad \tilde{\mathbf{f}}_n \in \mathbb{C}^{M \times 1}, \quad 1 \leq n \leq N.$$

Then, the algorithm of BFS method for solving (2.12) is given by

Algorithm 1 BFS for solving (2.12)

```
solve (w̃_0^(α) I_M + D_1 S) u_1 = f̃_1;
for k = 2 : N
    b = f̃_k;
    for i = 1 : k − 1
        b = b − w̃_{k−i}^(α) u_i;
    end
    solve (w̃_0^(α) I_M + D_k S) u_k = b;
end
```

In Algorithm 1, it requires to solve linear systems of the form

$$(\tilde{w}_0^{(\alpha)}\mathbf{I}_M + \mathbf{D}_n\mathbf{S})\mathbf{u}_n = \mathbf{y}, \tag{5.1}$$

with some given right hand side $\mathbf{y}$ and some $n \in \{1, 2, ..., N\}$. It is noticeable that the banded preconditioner proposed in [26] can be extended to precondition (5.1). Let

$$\mathbf{S}_b(k) = \frac{1}{h^\beta}\begin{bmatrix} s_0^{(\beta)} & \cdots & s_k^{(\beta)} & & \\ \vdots & s_0^{(\beta)} & \ddots & \ddots & \\ s_k^{(\beta)} & \ddots & \ddots & \ddots & s_k^{(\beta)} \\ & \ddots & \ddots & \ddots & \vdots \\ & & s_k^{(\beta)} & \cdots & s_0^{(\beta)} \end{bmatrix}_{M\times M} + \begin{bmatrix} a_{k1} & & & & \\ & a_{k2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & a_{kM} \end{bmatrix}, \tag{5.2}$$

with $a_{ki} = b_{ki} + b_{k,M-i}$ for all $1 \le i \le M$ and

$$b_{ki} = \begin{cases} 0, & 1 \le i \le k+1, \\ \frac{1}{h^\beta}\sum\limits_{j=k+1}^{i-1} s_j^{(\beta)}, & k+1 < i \le M. \end{cases}$$

Then, the banded preconditioner for (5.1) is given by $\mathbf{P}_n(k) = \tilde{w}_0^{(\alpha)}\mathbf{I}_M + \mathbf{D}_n\mathbf{S}_b(k)$ for $1 \le n \le N$. We solve (5.1) by the restarted version of PGMRES with preconditioner $\mathbf{P}_n(k)$ and the restarting number 50, where the stopping criterion is also set as $\frac{||\mathbf{r}_k||_2}{||\mathbf{r}_0||_2} \le 10^{-7}$. Meanwhile, since $\mathbf{u}_{n-1}$ is close to $\mathbf{u}_n$, we also set $\mathbf{u}_{n-1}$ as initial guess when using PGMRES to solve (5.1). Moreover, we denote by BFSP(k), a combination of Algorithm 1 and PGMRES inner solver with banded preconditioner $\mathbf{P}_n(k)$. To implement the BFSP(k) in an optimal way for fair comparison, we refer to the fast implementation presented in [26]. If there is no ambiguity, we also abbreviate it as BFSP.

Example 1. Consider the TSFDE (1.1)–(1.3) with

$$u(x,t) = t^2x^4(2-x)^4, \quad d(x,t) = 3 + \sin^2(20xt), \quad T = 1, \quad [a,b] = [0,2]$$
$$f(x,t) = \frac{2t^{2-\alpha}x^4(2-x)^4}{\Gamma(3-\alpha)} - \sigma_\beta d(x,t)t^2\sum_{i=5}^{9}\frac{q_i\Gamma(i)\left[x^{i-1-\beta} + (2-x)^{i-1-\beta}\right]}{\Gamma(i-\beta)},$$
$$q_5 = 16, \quad q_6 = -32, \quad q_7 = 24, \quad q_8 = -8, \quad q_9 = 1.$$

We solve Example 1 by the SEP and the BFSP(k) solver. The corresponding results are listed in Table 1–2.

Table 1: Results of the SEP preconditioner and the BFSP(k) solver when $M+1=2^{10}$.

		SEP			BFSP(8)		BFSP(12)		BFSP(16)	
(α,β)	N	iter	CPU	$\mathrm{E}_{N,M}$	CPU	$\mathrm{E}_{N,M}$	CPU	$\mathrm{E}_{N,M}$	CPU	$\mathrm{E}_{N,M}$
(0.1,1.1)	2^{12}	6	23.86s	1.22e-2	42.20s	1.22e-2	42.49s	1.22e-2	46.30s	1.22e-2
	2^{13}	6	48.29s	1.22e-2	124.63s	1.22e-2	124.69s	1.22e-2	130.47s	1.22e-2
	2^{14}	6	97.99s	1.22e-2	407.54s	1.22e-2	405.48s	1.22e-2	426.85s	1.22e-2
	2^{15}	6	207.62s	1.22e-2	1460.60s	1.22e-2	1428.30s	1.22e-2	1490.74s	1.22e-2
(0.1,1.9)	2^{12}	4	25.46s	3.65e-5	35.69s	3.63e-5	37.06s	3.63e-5	39.37s	3.63e-5
	2^{13}	4	48.32s	3.65e-5	112.83s	3.63e-5	113.16s	3.63e-5	120.52s	3.63e-5
	2^{14}	4	99.26s	3.65e-5	386.79s	3.63e-5	382.83s	3.63e-5	400.54s	3.63e-5
	2^{15}	4	210.21s	3.65e-5	1413.30s	3.63e-5	1463.79s	3.63e-5	1452.93s	3.63e-5
(0.5,1.5)	2^{12}	5	25.23s	1.10e-3	37.82s	1.10e-3	38.72s	1.10e-3	41.42s	1.10e-3
	2^{13}	5	47.93s	1.10e-3	116.17s	1.10e-3	116.96s	1.10e-3	123.08s	1.10e-3
	2^{14}	5	99.03s	1.10e-3	391.88s	1.10e-3	387.93s	1.10e-3	418.11s	1.10e-3
	2^{15}	5	210.70s	1.10e-3	1422.78s	1.10e-3	1410.74s	1.10e-3	1495.00s	1.10e-3
(0.9,1.1)	2^{12}	6	27.85s	1.11e-2	34.49s	1.11e-2	35.37s	1.11e-2	38.32s	1.11e-2
	2^{13}	6	54.40s	1.11e-2	110.54s	1.11e-2	109.61s	1.11e-2	118.85s	1.11e-2
	2^{14}	6	111.86s	1.11e-2	382.50s	1.11e-2	377.30s	1.11e-2	405.07s	1.11e-2
	2^{15}	6	235.68s	1.12e-2	1404.61s	1.11e-2	1435.99s	1.11e-2	1458.20s	1.11e-2
(0.9,1.9)	2^{12}	4	25.45s	2.59e-5	34.35s	2.54e-5	35.11s	2.54e-5	38.30s	2.54e-5
	2^{13}	4	48.48s	3.04e-5	110.13s	3.01e-5	112.82s	3.01e-5	117.20s	3.01e-5
	2^{14}	4	99.16s	3.26e-5	383.18s	3.23e-5	376.46s	3.27e-5	397.76s	3.85e-5
	2^{15}	4	210.45s	3.36e-5	1407.66s	4.02e-5	1420.45s	3.95e-5	1443.74s	3.91e-5

Table 2: Results of the SEP preconditioner and the BFSP(8) solver when $N = 2^{16}$.

(α, β)	$M+1$	SEP			BFSP(8)	
		iter	CPU	$\mathrm{E}_{N,M}$	CPU	$\mathrm{E}_{N,M}$
(0.1,1.1)	2^6	6	24.79s	1.60e-1	2205.40s	1.59e-1
	2^7	6	57.24s	8.85e-2	2347.66s	8.79e-2
	2^8	6	114.25s	4.68e-2	2855.14s	4.65e-2
	2^9	6	218.78s	2.41e-2	3743.70s	2.40e-2
(0.1,1.9)	2^6	6	21.72s	2.84e-4	1866.21s	2.83e-4
	2^7	4	45.90s	1.59e-4	2112.77s	1.58e-4
	2^8	4	101.82s	1.17e-4	2574.04s	1.17e-4
	2^9	4	221.59s	6.82e-5	3454.10s	6.79e-5
(0.5,1.5)	2^6	5	23.30s	1.69e-2	1860.77s	1.68e-2
	2^7	5	49.35s	8.70e-3	2119.97s	8.70e-3
	2^8	5	101.66s	4.40e-3	2538.51s	4.40e-3
	2^9	5	222.66s	2.20e-3	3420.24s	2.20e-3
(0.9,1.1)	2^6	5	23.34s	1.46e-1	1851.86s	1.46e-1
	2^7	5	49.18s	8.08e-2	2085.73s	8.02e-2
	2^8	6	100.49s	4.27e-2	2532.74s	4.24e-2
	2^9	6	218.72s	2.20e-2	3403.10s	2.18e-2
(0.9,1.9)	2^6	5	23.36s	2.72e-4	1847.91s	2.71e-4
	2^7	4	49.84s	1.46e-4	2142.08s	1.47e-4
	2^8	4	100.84s	1.10e-4	2576.41s	1.11e-4
	2^9	4	222.22s	6.38e-5	3470.37s	6.62e-5

In Table 1, we test different values of k for the BFSP(k) solver. As we see that, increasing the value of k does not improve the performance of BFSP(k), i.e., the CPU cost of the BFSP(k) solver is not decreasing as k increases. In addition, for large k, the BFSP(k) solver takes up more computer memory. Based on these considerations, we only test the BFSP(8) solver in Table 2. From Table 1–2, we see that the iteration number of the SEP solver always ranges from 4-6, which is bounded by a constant 6. The bounded iteration number implies that singular values of the preconditioned matrix are bounded above and below by mesh-parameters-independent positive constants, which is in accordance with our theoretical results. Moreover, the CPU cost of the SEP solver grows almost linearly as the matrix size increases, which implies that the complexity of our proposed method is nearly optimal, i.e., the complexity stays almost the same order as the number of unknowns. Also, by comparing the performance of the SEP and the BFSP solver, we see that the SEP solver is much more efficient than the BFSP solver, while the errors of the two solvers are comparable with each other.

6. Concluding Remarks

In this paper, we have considered linear systems arising from discretization of time-space fractional Caputo-Riesz diffusion equations. The coefficient matrix is a summation of block lower triangular Toeplitz matrix and a block diagonal matrix with each block being a diagonal-times-Toeplitz matrix. Preconditioning techniques for such matrices have never been studied or developed before. The main contribution of this paper is to propose the SEP preconditioner for these linear systems with such structure so that the Krylov subspace method for these preconditioned linear systems converges fast and independently on both temporal and spatial mesh parameters. Theoretically, we have shown that the singular values of the preconditioned matrix are bounded below and above by mesh-parameters-independent positive constants for spatially or temporally independent coefficients. Numerical experiments have been reported to show our proposed preconditioner is efficient. We will consider the analysis associated with a more general form of coefficients and nonlinear time-space fractional equations as our future research works.

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