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A SPLITTING PRECONDITIONER FOR TOEPLITZ-LIKE LINEAR SYSTEMS ARISING FROM FRACTIONAL DIFFUSION EQUATIONS*

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Abstract. In this paper, we study Toeplitz-like linear systems arising from time-dependent one-dimensional and two-dimensional Riesz space-fractional diffusion equations with variable diffusion coefficients. The coefficient matrix is a sum of a scalar identity matrix and a diagonal-times-Toeplitz matrix which allows fast matrix-vector multiplication in iterative solvers. We propose and develop a splitting preconditioner for this kind of matrix and analyze the spectra of the preconditioned matrix. Under mild conditions on variable diffusion coefficients, we show that the singular values of the preconditioned matrix are bounded above and below by positive constants which are independent of temporal and spatial discretization step-sizes. When the preconditioned conjugate gradient squared method is employed to solve such preconditioned linear systems, the method converges linearly within an iteration number independent of the discretization step-sizes. Numerical examples are given to illustrate the theoretical results and demonstrate that the performance of the proposed preconditioner is better than other tested solvers.

Key words. diagonal-times-Toeplitz matrices, preconditioners, variable coefficients, space-fractional diffusion equations, Krylov subspace methods

AMS subject classifications. 65B99, 65M22, 65F08, 65F10

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1. Introduction.

We first consider a one-dimensional initial-boundary value problem of space-fractional diffusion equation [9] (the two-dimensional case will be considered in section 3):

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), \qquad (x,t) \in (x_L, x_R) \times (0,T], \tag{1}$$

$$u(x_L,t) = u(x_R,t) = 0, \qquad t \in (0,T], \tag{2}$$

$$u(x,0) = \psi(x), \qquad x \in [x_L, x_R], \tag{3}$$

where the coefficient $d(x,t)$ is larger than a positive constant, $u(x,t)$ is unknown to be solved, $f(x,t)$ is the source term, and $\psi(x)$ is the initial condition. $\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}$ is the Riesz fractional derivative of order $\alpha \in (1,2)$ with respect to x, whose definition is given by [9]

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} := \sigma_\alpha \left({}_{x_L}D_x^\alpha + {}_xD_{x_R}^\alpha \right) u(x,t), \quad (x,t) \in (x_L, x_R) \times (0,T], \tag{4}$$

where $\sigma_\alpha = -\frac{1}{2\cos(\frac{\pi\alpha}{2})} > 0$, and ${}_{x_L}D_x^\alpha u(x,t)$ and ${}_xD_{x_R}^\alpha u(x,t)$ are the left- and right-sided Riemann–Liouville (RL) derivatives, respectively, with their definitions given as

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follows [8]:

$$ {}_{x_L}D_x^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{x_L}^{x} \frac{u(\xi,t)}{(x-\xi)^{\alpha-1}} d\xi $$

and

$$ {}_{x}D_{x_R}^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{x}^{x_R} \frac{u(\xi,t)}{(\xi-x)^{\alpha-1}} d\xi. $$

Here, $\Gamma(\cdot)$ denotes the gamma function.

In the last few decades, fractional calculus including fractional differentiation and integration has gained considerable attention and importance due to its applications in various fields of science and engineering, such as electrical and mechanical engineering, biology, physics, control theory, and data fitting; see [25, 13, 5, 21, 26, 1]. As a class of fractional differential equations, fractional diffusion equations have been widely and successfully used in modeling challenging phenomena such as long-range interactions and nonlocal dynamics [4, 25].

Since the closed-form analytical solutions of fractional diffusion equations are usually unavailable, many discretization schemes are proposed in order to provide more systematic ways to solve fractional diffusion equations; see, for instance, [12, 31, 16, 19, 20, 27, 30, 8, 6]. Nevertheless, since the Riesz fractional differential operator is nonlocal, its numerical discretization leads to dense matrices. That means the direct solver for the linear systems arising from discretization of fractional diffusion equations requires very high computational complexity when the grid is dense. This motivates us to develop fast solvers for linear systems arising from fractional diffusion equations.

For implicit uniform-grid discretization of fractional diffusion equation (1)–(3), it requires solving a Toeplitz-like linear system whose coefficient matrix is a summation of a scalar identity and a diagonal-times-one-level-Toeplitz matrix at each time step. In general, there is no fast direct solver for this kind of linear system. Fortunately, the Toeplitz-like structure allows fast matrix-vector multiplication. In [24, 17, 23, 14], iterative solvers are studied for linear systems arising from fractional diffusion equations. Their theoretical results are established either under the assumption that $d(x,t)$ is a constant [17, 24] or under the assumption that the ratio between temporal and spatial discretization parameters, τ/h^{α}, is a constant [23, 10].

The main contribution of this paper is to propose a new preconditioner for Toeplitz-like linear systems arising from fractional diffusion equations with variable diffusion coefficients. Our idea is to develop a splitting preconditioner for this kind of matrix by decomposing it into two matrix components: one is a diagonal matrix containing the variable diffusion coefficients, and the other is a Toeplitz matrix containing the discretization of the Riesz fractional derivative. This splitting strategy allows us to compute the inverse of the preconditioner very efficiently. Theoretically, we show that the singular values of the preconditioned matrix are bounded above and below by positive constants independent of temporal and spatial discretization step-sizes under two assumptions: (a) $d(x,t)$ is strictly positive, is Lipschitz continuous with respect to x, and has a Lipschitz constant independent of t; (b) the discretization matrix to the operator $-\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}$ is symmetric positive definite and has polynomial decay with a decay order of $\alpha+1$. Assumption (a) allows $d(x,t)$ to be nonconstant without a restriction on the ratio τ/h^{α}. Hence, compared with assumptions in [24, 17, 23, 14, 10] as mentioned above, assumption (a) is a relatively mild one. Besides, we verify a series of discretization schemes in section 4 to show that assumption (b) is easily satisfied. On the other hand, because of the uniformly bounded singular values of the preconditioned matrix under assumptions (a) and (b), the condition number of the preconditioned matrix is bounded by a constant independent of discretization step-sizes. When the conjugate gradient squared method is employed to solve such a preconditioned linear system, the method converges linearly within an iteration number independent of discretization step-sizes. The proposed preconditioning technique and analysis can be extended to time-dependent two-dimensional Riesz fractional diffusion equations. Numerical examples are given to illustrate the theoretical results and demonstrate that the performance of the proposed preconditioner is better than other tested solvers.

The outline of this paper is as follows. In section 2, we study the new splitting preconditioner and analyze singular values of the preconditioned matrix. In section 3, we extend the preconditioner to two-dimensional fractional diffusion equations. In section 4, we verify a series of discretization schemes for $-\frac{\partial^\alpha}{\partial|x|^\alpha}$ and show that these schemes satisfy the assumption required in previous sections. In section 5, we present numerical results to show the performance of the proposed preconditioner. Finally, we give concluding remarks in section 6.

2. The splitting preconditioner.

2.1. Toeplitz-like discretization systems. In this subsection, we present implicit uniform-grid discretization of the one-dimensional space fractional diffusion equation and the resulting Toeplitz-like linear systems. For positive integers M and N, let $\tau = T/N$ and $h = (x_R - x_L)/(M+1)$. Define the temporal grid and the spatial grid, respectively, by $\{t_n | t_n = n\tau,\ 0 \le n \le N\}$ and $\{x_i | x_i = x_L + ih,\ 0 \le i \le M+1\}$. Also, we let $\mathbf{x} = (x_1, x_2, \ldots, x_M)^{\mathrm{T}}$, $d_{i,n} = d(x_i, t_n)$ for $0 \le i \le M+1$ and $0 \le n \le N$. Without loss of generality, we assume that uniform-grid discretization of the Riesz fractional derivative is given by (see, e.g., [9, 28, 6])

$$\left.\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha}\right|_{(x,t)=(x_i,t_n)} \approx -\frac{1}{h^\alpha}\sum_{j=1}^{M} s_{|i-j|}^{(\alpha)} u(x_j, t_n), \quad 1 \le i \le M, \quad 0 \le n \le N, \tag{5}$$

where $s_k^{(\alpha)} (k \ge 0)$ are real numbers varying from different discretization schemes. A series of choices of $\{s_k^{(\alpha)}\}_{k\ge 0}$ will be discussed in section 4. By using (5) and the backward difference approximation to $\frac{\partial u}{\partial t}$, we obtain an implicit difference discretization of the stochastic functional differential equation (SFDE) in (1)–(3) as follows:

$$\tau^{-1}(\mathbf{u}_{n+1} - \mathbf{u}_n) = -h^{-\alpha}\mathbf{D}_n \mathbf{S}_\alpha \mathbf{u}_{n+1} + \mathbf{f}_{n+1}, \quad 0 \le n \le N-1, \tag{6}$$

where $\mathbf{u}_0 = \psi(\mathbf{x})$, $\mathbf{u}_n$ is the approximate solution to $u(\mathbf{x}, t_n)$, $\mathbf{f}_n = f(\mathbf{x}, t_n)$ for $1 \le n \le N$,

$$\mathbf{D}_n = \mathrm{diag}(d_{1,n}, d_{2,n}, \ldots, d_{M,n}), \quad 1 \le n \le N, \quad \text{and}$$

$$\mathbf{S}_\alpha = \begin{bmatrix} s_0^{(\alpha)} & s_1^{(\alpha)} & \cdots & s_{M-2}^{(\alpha)} & s_{M-1}^{(\alpha)} \\ s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} & \cdots & s_{M-2}^{(\alpha)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{M-2}^{(\alpha)} & \cdots & s_1^{(\alpha)} & s_0^{(\alpha)} & s_1^{(\alpha)} \\ s_{M-1}^{(\alpha)} & s_{M-2}^{(\alpha)} & \cdots & s_1^{(\alpha)} & s_0^{(\alpha)} \end{bmatrix}.$$

To solve (6) is equivalent to solving one after another the following Toeplitz-like linear systems:

$$\mathbf{A}_n\mathbf{u}_n = \mathbf{b}_n, \qquad 1 \le n \le N, \tag{7}$$

where

$$\mathbf{A}_n = \mathbf{I}_M + \eta\mathbf{D}_n\mathbf{S}_\alpha, \quad \eta = \tau/h^\alpha, \qquad \mathbf{b}_n = \mathbf{u}_{n-1} + \tau\mathbf{f}_n,$$

$\mathbf{I}_M$ denotes an $M\times M$ identity matrix, and $\mathbf{A}_n$ is a Toeplitz-like matrix. We note that the matrix-vector multiplication of $\mathbf{A}_n$ can be done fast with $\mathcal{O}(M\log M)$ operation and $\mathcal{O}(M)$ storage via using fast Fourier transformations (FFTs); see [22]. To make it clear, $\mathbf{S}_\alpha$ is assumed to be symmetric positive definite for any $\alpha \in (1,2)$ in this paper, which is essential in both the theoretical and numerical senses. In section 4, we will show that a series of existing discretization schemes satisfies this assumption.

In this paper, we propose using a splitting preconditioner for (7). For a given diagonal matrix $\mathbf{\Phi}$, denote the average of its diagonal entries by $\mathtt{mean}(\mathbf{\Phi})$. The proposed preconditioner is the product of two matrices:

$$\mathbf{P}_n = \mathbf{W}_n\mathbf{T}_n, \quad 1 \le n \le N, \tag{8}$$

where $\mathbf{W}_n = \mathbf{D}_n + \mathbf{I}_M$ is a diagonal matrix containing the discretization of variable diffusion coefficients, $\mathbf{T}_n = \bar{\theta}_n\mathbf{I}_M + \eta\bar{d}_n\mathbf{S}_\alpha$ is a Toeplitz matrix containing the discretization of the Riesz fractional derivative with $\bar{\theta}_n = \mathtt{mean}(\mathbf{W}_n^{-1})$, and $\bar{d}_n = \mathtt{mean}(\mathbf{D}_n\mathbf{W}_n^{-1})$. It is obvious that when diffusion coefficients are constant, $\mathbf{D}_n$ is just a scalar-times-identity matrix and the proposed preconditioner is exactly the coefficient matrix $\mathbf{A}_n$ itself.

2.2. The spectral properties. In this subsection, we study the singular values of the preconditioned matrices, $\mathbf{A}_n\mathbf{P}_n^{-1}$, for each $1 \le n \le N$.

An essential property of $\mathbf{P}_n$ is its invertibility.

PROPOSITION 2.1. *$\mathbf{P}_n$ given in* (8) *is invertible for each n.*

Proof. Since the diagonal entries of $\mathbf{W}_n (1 \le n \le N)$ are all nonzeros, $\mathbf{W}_n$ are invertible. Moreover, $\bar{\theta}_n$ must be positive. Note also that $\mathbf{S}_\alpha$ is real symmetric positive definite for several numerical schemes (see section 4). Thus, $\mathbf{T}_n = \bar{\theta}_n\mathbf{I}_M + \eta\bar{d}_n\mathbf{S}_\alpha$ is real symmetric positive definite. The result follows. ∎

Before analyzing the singular values, we first introduce several lemmas and notations. Define a set of sequences as follows:

$$\mathcal{D}_s := \left\{ \{w_k\}_{k\ge 0} \;\middle|\; ||\{w_k\}||_{\mathcal{D}_s} := \sup_{k\ge 0} |w_k|(1+k)^{1+s} < +\infty \right\}$$

for some $s > 0$. Then, it is easy to check that $\mathcal{D}_s$ is a linear normed space equipped with norm $||\cdot||_{\mathcal{D}_s}$. We denote by $\mathbb{R}^{m\times n}$ the set of all $m\times n$ real matrices. For a nonnegative diagonal matrix $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_m) \in \mathbb{R}^{m\times m}$, we are interested in the following parameters:

$$\mathtt{range}(\mathbf{D}) = \left[\min_{1\le i\le m} d_i, \max_{1\le i\le m} d_i\right], \quad \min(\mathbf{D}) = \min_{1\le i\le m} d_i,$$

$$\nabla(\mathbf{D}) := \max_{1\le i,j\le m, i\ne j} \frac{|d_i - d_j|}{|i-j|}, \quad \Delta_{\mathbf{S}_\alpha}(\mathbf{D}) \equiv \mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D} - 2\mathbf{D}^{\frac{1}{2}}\mathbf{S}_\alpha\mathbf{D}^{\frac{1}{2}}.$$

The following three lemmas are used to establish our main results in Theorem 5.

Lemma 2.2. *Let* $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_M) \in \mathbb{R}^{M\times M}$. *Assume*
(i) $\min(\mathbf{D}) \geq \check{d} > 0$,
(ii) $\nabla(\mathbf{D}) \leq \tilde{d}h$, *and*
(iii) $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \leq c_0$.
Then, $||\Delta_{\mathbf{S}_\alpha}(\mathbf{D})||_2 \leq \mu(c_0, \tilde{d}, \check{d})h^\alpha$, *where*

$$\mu(x,y,z) = \frac{xy^2(x_R - x_L)^{2-\alpha}}{2z(2-\alpha)}, \qquad x, y \geq 0, \text{ and } z > 0. \tag{9}$$

Proof. It is easy to check that the (i,j)th entry $r_{i,j}$ of $\Delta_{\mathbf{S}_\alpha}(\mathbf{D})$ is given by

$$r_{i,j} = d_i s^{(\alpha)}_{|i-j|} + s^{(\alpha)}_{|i-j|} d_j - 2d_i^{\frac{1}{2}} s^{(\alpha)}_{|i-j|} d_j^{\frac{1}{2}} = (d_i^{\frac{1}{2}} - d_j^{\frac{1}{2}})^2 s^{(\alpha)}_{|i-j|}, \qquad 1 \leq i, j \leq M.$$

Using assumptions (ii)–(iii), it holds that

$$\begin{aligned} |r_{i,j}| = \left|s^{(\alpha)}_{|i-j|}\right|\left|d_i^{\frac{1}{2}} - d_j^{\frac{1}{2}}\right|^2 = \left|s^{(\alpha)}_{|i-j|}\right|\left|\int_{d_i}^{d_j} \frac{1}{2}\xi^{-\frac{1}{2}} d\xi\right|^2 &\leq \left|s^{(\alpha)}_{|i-j|}\right|\left|\int_{d_i}^{d_j} \frac{1}{2}\check{d}^{-\frac{1}{2}} d\xi\right|^2 \\ &= 4^{-1}\check{d}^{-1}|s^{(\alpha)}_{|i-j|}||d_i - d_j|^2 \\ &\leq 4^{-1}\check{d}^{-1}|s^{(\alpha)}_{|i-j|}|\tilde{d}^2 h^2 |i-j|^2 \\ &\leq \frac{c_0\tilde{d}^2h^2|i-j|^2}{4\check{d}(1+|i-j|)^{\alpha+1}}. \end{aligned} \tag{10}$$

Using (10), we obtain

$$\begin{aligned} ||\Delta_{\mathbf{S}_\alpha}(\mathbf{D})||_\infty &= \max_{1\leq i\leq M}\left(|r_{ii}| + \sum_{j=1}^{i-1}|r_{ij}| + \sum_{j=i+1}^{M}|r_{ij}|\right) \\ &\leq \max_{1\leq i\leq M}\frac{c_0\tilde{d}^2h^2}{4\check{d}}\left(\sum_{j=1}^{i-1}\frac{|i-j|^2}{(1+|i-j|)^{\alpha+1}} + \sum_{j=i+1}^{M}\frac{|i-j|^2}{(1+|i-j|)^{\alpha+1}}\right) \\ &= \max_{1\leq i\leq M}\frac{\tilde{d}^2c_0h^2}{4\check{d}}\left(\sum_{k=1}^{i-1}\frac{k^2}{(1+k)^{\alpha+1}} + \sum_{k=1}^{M-i}\frac{k^2}{(1+k)^{\alpha+1}}\right) \\ &\leq \frac{\tilde{d}^2c_0h^2}{2\check{d}}\sum_{k=1}^{M}k^{1-\alpha} \\ &\leq \frac{\tilde{d}^2c_0h^2}{2\check{d}}\sum_{k=1}^{M}\int_{k-1}^{k}x^{1-\alpha}dx \\ &= \frac{c_0\tilde{d}^2h^\alpha M^{2-\alpha}}{2\check{d}(2-\alpha)}\left(\frac{x_R - x_L}{M+1}\right)^{2-\alpha} \leq \mu(c_0, \tilde{d}, \check{d})h^\alpha. \end{aligned}$$

Since $\mathbf{S}_\alpha$ is symmetric, $\Delta_{\mathbf{S}_\alpha}(\mathbf{D})$ is also symmetric. Therefore, we have $||\Delta_{\mathbf{S}_\alpha}(\mathbf{D})||_2 = \rho(\Delta_{\mathbf{S}_\alpha}(\mathbf{D})) \leq ||\Delta_{\mathbf{S}_\alpha}(\mathbf{D})||_\infty$, which completes the proof. ∎

For any symmetric matrices $\mathbf{H}_1, \mathbf{H}_2 \in \mathbb{R}^{m\times m}$, denote $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{H}_1$ if $\mathbf{H}_2 - \mathbf{H}_1$ is symmetric positive (or semidefinite) definite. Especially, we denote $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{O}$ if $\mathbf{H}_2$ itself is symmetric positive (or semidefinite) definite. Also, $\mathbf{H}_1 \prec$ (or $\preceq$) $\mathbf{H}_2$

and $\mathbf{O} \prec$ (or $\preceq$)$\mathbf{H}_2$ have the same meanings as those of $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{O}$ and $\mathbf{H}_2 \succ$ (or $\succeq$) $\mathbf{O}$, respectively.

LEMMA 2.3. *Let* $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_M) \in \mathbb{R}^{M\times M}$. *Assume*

(i) $\mathtt{range}(\mathbf{D}) \subset [\check{d}, \hat{d}]$ *with* $\check{d} > 0$,

(ii) $\nabla(\mathbf{D}) \le \tilde{d}h$, *and*

(iii) $\{s_k^{(\alpha)}\}_{k\ge 0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \le c_0$.

Then, $\check{d}\mathbf{S}_\alpha - 2\mu(c_0, \tilde{d}, \check{d})h^\alpha \mathbf{I}_M \prec \mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D} \prec 4\hat{d}\mathbf{S}_\alpha + \mu(c_0, \tilde{d}, \hat{d})h^\alpha\mathbf{I}_M$, *where the function* $\mu(\cdot,\cdot,\cdot)$ *is defined in* (9).

Proof. We note that $\nabla(\cdot)$ is shift-invariant. i.e.,

$$\nabla(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M) = \nabla(\mathbf{D}) \le \tilde{d}h. \tag{11}$$

Moreover, $\min(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M) \ge 2^{-1}\check{d}$. By Lemma 2.2, it holds that

$$||\Delta_{\mathbf{S}_\alpha}(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)||_2 \le \mu(c_0, \tilde{d}, 2^{-1}\check{d})h^\alpha = 2\mu(c_0, \tilde{d}, \check{d})h^\alpha.$$

Moreover, $\mathbf{O} \prec (\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)^{\frac{1}{2}}$ and $\mathbf{O} \prec \mathbf{S}_\alpha$ imply that $\mathbf{O} \prec (\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)^{\frac{1}{2}}\mathbf{S}_\alpha(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)^{\frac{1}{2}}$. Hence,

$$\begin{aligned}\mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D} &= \check{d}\mathbf{S}_\alpha + (\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)\mathbf{S}_\alpha + \mathbf{S}_\alpha(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)\\ &= \check{d}\mathbf{S}_\alpha + 2(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)^{\frac{1}{2}}\mathbf{S}_\alpha(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)^{\frac{1}{2}} + \Delta_{\mathbf{S}_\alpha}(\mathbf{D} - 2^{-1}\check{d}\mathbf{I}_M)\\ &\succ \check{d}\mathbf{S}_\alpha - 2\mu(c_0, \tilde{d}, \check{d})h^\alpha\mathbf{I}_M.\end{aligned} \tag{12}$$

Again, by shift-invariance of $\nabla(\cdot)$, $\nabla(2\hat{d}\mathbf{I}_M - \mathbf{D}) = \nabla(\mathbf{D}) \le \tilde{d}h$. Besides, (i) also yields that $\min(2\hat{d}\mathbf{I}_M - \mathbf{D}) \ge \hat{d}$. By Lemma 2.2 again,

$$||\Delta_{\mathbf{S}_\alpha}(2\hat{d}\mathbf{I}_M - \mathbf{D})||_2 \le \mu(c_0, \tilde{d}, \hat{d})h^\alpha.$$

Moreover, $\mathbf{O} \prec (2\hat{d}\mathbf{I}_M - \mathbf{D})^{\frac{1}{2}}$ and $\mathbf{O} \prec \mathbf{S}_\alpha$ imply that $\mathbf{O} \prec (2\hat{d}\mathbf{I}_M - \mathbf{D})^{\frac{1}{2}}\mathbf{S}_\alpha(2\hat{d}\mathbf{I}_M - \mathbf{D})^{\frac{1}{2}}$. Hence,

$$\begin{aligned}4\hat{d}\mathbf{S}_\alpha &= (2\hat{d}\mathbf{I}_M - \mathbf{D})\mathbf{S}_\alpha + \mathbf{S}_\alpha(2\hat{d}\mathbf{I}_M - \mathbf{D}) + \mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D}\\ &= \mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D} + 2(2\hat{d}\mathbf{I}_M - \mathbf{D})^{\frac{1}{2}}\mathbf{S}_\alpha(2\hat{d}\mathbf{I}_M - \mathbf{D})^{\frac{1}{2}} + \Delta_{\mathbf{S}_\alpha}(2\hat{d}\mathbf{I}_M - \mathbf{D})\\ &\succ \mathbf{D}\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D} - \mu(c_0, \tilde{d}, \hat{d})h^\alpha\mathbf{I}_M.\end{aligned} \tag{13}$$

The result follows from (12) and (13). ∎

PROPOSITION 2.4. *For positive numbers* ξ_i, ζ_i $(1 \le i \le m)$, *it obviously holds that*

$$\min_{1\le i\le m}\frac{\xi_i}{\zeta_i} \le \left(\sum_{i=1}^m \zeta_i\right)^{-1}\left(\sum_{i=1}^m \xi_i\right) \le \max_{1\le i\le m}\frac{\xi_i}{\zeta_i}.$$

For $\mathbf{C} \in \mathbb{R}^{m\times n}$, let $\Sigma(\mathbf{C})$ denote the set of singular values of $\mathbf{C}$. Also denote $\Sigma^2(\mathbf{C}) = \{\sigma^2 | \sigma \in \Sigma(\mathbf{C})\}$. For any invertible matrix, $\mathbf{C} \in \mathbb{R}^{m\times m}$, define its condition number as

$$\mathrm{cond}(\mathbf{C}) \triangleq ||\mathbf{C}||_2||\mathbf{C}^{-1}||_2.$$

For a given domain Ω, define the set of all Lipschitz continuous functions on Ω as

$$\mathcal{L}(\Omega) := \left\{ v(x) \;\middle|\; |v|_{\mathcal{L}(\Omega)} := \sup_{x,y\in\Omega, x\neq y} \frac{|v(x)-v(y)|}{|x-y|} < +\infty \right\}.$$

THEOREM 2.5. *Assume*
(i) $\forall (x,t) \in (x_L, x_R) \times (0,T]$, $d(x,t) \in [\check{d}, \hat{d}]$ *with* $\check{d} > 0$,
(ii) $\forall t \in (0,T]$, $d(\cdot,t) \in \mathcal{L}((x_L,x_R))$ *with*

$$\sup_{t\in(0,T]} |d(\cdot,t)|_{\mathcal{L}((x_L,x_R))} \leq \tilde{d}, \text{ and}$$

(iii) $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \leq c_0$.
Then, for any $N \geq N_0$, $\bigcup_{n=1}^{N} \Sigma^2(\mathbf{A}_n\mathbf{P}_n^{-1}) \subset [\check{s}, \hat{s}]$, *and thus*

$$\sup_{M\geq 1} \sup_{N\geq N_0} \max_{1\leq n\leq N} \operatorname{cond}(\mathbf{A}_{\rm n}\mathbf{P}_{\rm n}^{-1}) \leq \sqrt{\hat{\rm s}/\check{\rm s}},$$

where $\check{s}$, $\hat{s}$, *and* N_0 *are positive constants independent of* τ *and* h*:*

$$\check{s} = \min\left\{ \frac{\check{d}(\check{d}+1)}{4\hat{d}(\hat{d}+1)}, \frac{\check{d}^2}{\hat{d}^2} \right\}, \quad \hat{s} = \frac{1}{\check{s}},$$
$$N_0 = 4T \max\left\{ \mu(c_0, \tilde{d}, \check{d}), 4\hat{d}\mu(c_0, (1+\hat{d})\tilde{d}, (1+\check{d})^4) \right\},$$

and $\mu(\cdot,\cdot,\cdot)$ *is defined in* (9).

Proof. By assumption (i), $\texttt{range}(\mathbf{D}_n) \subset [\check{d}, \hat{d}]$. By assumption (ii),

$$\nabla(\mathbf{D}_n) = \max_{1\leq i,j\leq M, i\neq j} \frac{|d(x_i,t_n) - d(x_j,t_n)|}{|i-j|} \leq \max_{1\leq i,j\leq M, i\neq j} \frac{\tilde{d}|x_i - x_j|}{|i-j|} = \tilde{d}h.$$

Hence, by Lemma 2.3, we have

$$\check{d}\mathbf{S}_\alpha - 2\mu(c_0, \tilde{d}, \check{d})h^\alpha \mathbf{I}_M \prec \mathbf{S}_\alpha \mathbf{D}_n + \mathbf{D}_n \mathbf{S}_\alpha \prec 4\hat{d}\mathbf{S}_\alpha + \mu(c_0, \tilde{d}, \hat{d})h^\alpha \mathbf{I}_M.$$

For $N \geq N_0$, we obtain

$$\tau = \frac{T}{N} \leq \frac{T}{N_0} = \frac{1}{4} \min\left\{ \frac{1}{\mu(c_0, \tilde{d}, \check{d})}, \frac{1}{4\hat{d}\mu(c_0, (1+\hat{d})\tilde{d}, (1+\check{d})^4)} \right\}. \tag{14}$$

Equation (14) implies

$$\tau \leq \frac{1}{4\mu(c_0, \tilde{d}, \check{d})} \leq \frac{1}{4\mu(c_0, \tilde{d}, \hat{d})}.$$

Therefore, $\eta\check{d}\mathbf{S}_\alpha - 2^{-1}\mathbf{I}_M \prec \eta(\mathbf{S}_\alpha\mathbf{D}_n + \mathbf{D}_n\mathbf{S}_\alpha) \prec 4\eta\hat{d}\mathbf{S}_\alpha + 4^{-1}\mathbf{I}_M$, which together with the fact that $\mathbf{A}_n^{\rm T}\mathbf{A}_n = \mathbf{I}_M + \eta(\mathbf{D}_n\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D}_n) + \eta^2\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha$ implies

$$\mathbf{O} \prec 2^{-1}\mathbf{I}_M + \eta\check{d}\mathbf{S}_\alpha + \eta^2\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha \prec \mathbf{A}_n^{\rm T}\mathbf{A}_n \prec (5/4)\mathbf{I}_M + 4\eta\hat{d}\mathbf{S}_\alpha + \eta^2\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha. \tag{15}$$

Denote $\check{w} = (1+\check{d})$ and $\hat{w} = (1+\hat{d})$. By assumption (i), we know that $\texttt{range}(\mathbf{W}_n^2) \subset [\check{w}^2, \hat{w}^2]$. Moreover,

$$\nabla(\mathbf{W}_n^2) = \max_{1\le i,j\le M, i\ne j} \frac{|(1+d_{i,n})^2 - (1+d_{j,n})^2|}{|i-j|} \le \max_{1\le i,j\le M, i\ne j} \frac{(1+d_{i,n}+1+d_{j,n})|d_{i,n}-d_{j,n}|}{|i-j|} \le 2\hat{w}\tilde{d}h.$$

Hence, by Lemma 2.3 again,

$$\check{w}^2\mathbf{S}_\alpha - 2\mu(c_0, 2\hat{w}\tilde{d}, \check{w}^2)h^\alpha \mathbf{I}_M \prec \mathbf{S}_\alpha\mathbf{W}_n^2 + \mathbf{W}_n^2\mathbf{S}_\alpha \prec 4\hat{w}^2\mathbf{S}_\alpha + \mu(c_0, 2\hat{w}\tilde{d}, \hat{w}^2)h^\alpha \mathbf{I}_M.$$

By (14),

$$\tau \le \frac{1}{16\hat{d}\mu(c_0, (1+\hat{d})\tilde{d}, (1+\check{d})^4)} = \frac{\check{w}^2}{4\hat{d}\mu(c_0, 2\hat{w}\hat{d}, \check{w}^2)} \le \frac{\hat{w}^2}{4\hat{d}\mu(c_0, 2\hat{w}\hat{d}, \hat{w}^2)}.$$

Hence,

$$\eta\bar{d}_n\check{w}^2\mathbf{S}_\alpha - 2^{-1}\hat{d}^{-1}\bar{d}_n\check{w}^2\mathbf{I}_M \prec \eta\bar{d}_n(\mathbf{S}_\alpha\mathbf{W}_n^2 + \mathbf{W}_n^2\mathbf{S}_\alpha) \prec 4\eta\bar{d}_n\hat{w}^2\mathbf{S}_\alpha + 4^{-1}\hat{d}^{-1}\bar{d}_n\hat{w}^2\mathbf{I}_M,$$

which together with the fact that $\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n = \bar{\theta}_n^2\mathbf{W}_n^2 + \eta\bar{d}_n\bar{\theta}_n(\mathbf{W}_n^2\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{W}_n^2) + \eta^2\bar{d}_n^2\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha$ implies

$$\begin{cases} \mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n \succ \bar{\theta}_n\Big(\bar{\theta}_n\mathbf{W}_n^2 - \frac{\bar{d}_n\check{w}^2}{2\hat{d}}\mathbf{I}_M + \eta\bar{d}_n\check{w}^2\mathbf{S}_\alpha\Big) + \eta^2\bar{d}_n^2\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha, \\ \mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n \prec \bar{\theta}_n\Big(\bar{\theta}_n\mathbf{W}_n^2 + \frac{\bar{d}_n\hat{w}^2}{4\hat{d}}\mathbf{I}_M + 4\eta\bar{d}_n\hat{w}^2\mathbf{S}_\alpha\Big) + \eta^2\bar{d}_n^2\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha. \end{cases}$$

By using assumption (i),

$$\frac{1}{\hat{w}} \le \bar{\theta}_n \le \frac{1}{\check{w}}, \qquad \check{w}^2\mathbf{I}_M \preceq \mathbf{W}_n^2 \preceq \hat{w}^2\mathbf{I}_M, \qquad \frac{\check{d}}{\check{w}} \le \bar{d}_n = \frac{1}{M}\sum_{i=1}^{M}\frac{d_{i,n}}{1+d_{i,n}} \le \frac{\hat{d}}{\hat{w}}.$$

Hence,

$$\mathbf{O} \prec \frac{\check{w}^2}{2\hat{w}^2}\mathbf{I}_M + \frac{\eta\check{d}\check{w}}{\hat{w}}\mathbf{S}_\alpha + \frac{\eta^2\check{d}^2}{\check{w}^2}\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha \prec \mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n \prec \frac{5\hat{w}^2}{4\check{w}^2}\mathbf{I}_M + \frac{4\eta\hat{d}\hat{w}}{\check{w}}\mathbf{S}_\alpha + \frac{\eta^2\hat{d}^2}{\hat{w}^2}\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha. \tag{16}$$

For any nonzero vector $\mathbf{y} \in \mathbb{R}^{M\times 1}$, denote $\mathbf{z} = \mathbf{P}_n^{-1}\mathbf{y}$. Then, it holds that

$$\frac{\mathbf{y}^{\mathrm{T}}(\mathbf{A}_n\mathbf{P}_n^{-1})^{\mathrm{T}}(\mathbf{A}_n\mathbf{P}_n^{-1})\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} = \frac{\mathbf{z}^{\mathrm{T}}\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{z}}.$$

By (i), it is easy to check that

$$\frac{\check{d}^2}{\hat{w}^2} \le \frac{\mathbf{z}^{\mathrm{T}}\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha\mathbf{z}} \le \frac{\hat{d}^2}{\check{w}^2}.$$

Hence, applying Proposition 2.4 to (15) and (16) yields

$$\begin{aligned}\check{s} = \min\left\{\frac{2\check{w}^2}{5\hat{w}^2}, \frac{\check{d}\check{w}}{4\hat{d}\hat{w}}, \frac{\check{d}^2}{\hat{d}^2}\right\} &\le \frac{\mathbf{z}^{\mathrm{T}}(2^{-1}\mathbf{I}_M + \eta\check{d}\mathbf{S}_\alpha + \eta^2\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha)\mathbf{z}}{\mathbf{z}^{\mathrm{T}}[\frac{5\hat{w}^2}{4\check{w}^2}\mathbf{I}_M + \frac{4\eta\hat{d}\hat{w}}{\check{w}}\mathbf{S}_\alpha + \frac{\eta^2\hat{d}^2}{\hat{w}^2}\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha]\mathbf{z}} \\ &\le \frac{\mathbf{z}^{\mathrm{T}}\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{z}} \\ &\le \frac{\mathbf{z}^{\mathrm{T}}[(5/4)\mathbf{I}_M + 4\eta\hat{d}\mathbf{S}_\alpha + \eta^2\mathbf{S}_\alpha\mathbf{D}_n^2\mathbf{S}_\alpha]\mathbf{z}}{\mathbf{z}^{\mathrm{T}}[\frac{\check{w}^2}{2\hat{w}^2}\mathbf{I}_M + \frac{\eta\check{d}\check{w}}{\hat{w}}\mathbf{S}_\alpha + \frac{\eta^2\check{d}^2}{\check{w}^2}\mathbf{S}_\alpha\mathbf{W}_n^2\mathbf{S}_\alpha]\mathbf{z}} \\ &\le \max\left\{\frac{5\hat{w}^2}{2\check{w}^2}, \frac{4\hat{d}\hat{w}}{\check{d}\check{w}}, \frac{\hat{d}^2}{\check{d}^2}\right\} = \hat{s}.\end{aligned}$$

The results follow. ∎

Remark 2.6. Theorem 2.5 shows that the preconditioned matrix $\mathbf{A}_n\mathbf{P}_n^{-1}$ has a uniformly bounded condition number independent of τ and h under the related assumptions. On the other hand, it is numerically illustrated in Table 5 of section 5 that $\mathbf{A}_n$ has a condition number almost linearly dependent on η, which is ill-conditioned for large η. Thus, our splitting preconditioning technique improves the condition number of $\mathbf{A}_n$ and allows us to solve more efficiently the corresponding linear system for different values of the discretization parameters τ and h. According to Theorem 2.5, when the conjugate gradient method is employed to solve the normalized preconditioned system, the method converges linearly within an iteration number independent of τ and h.

3. Two-dimensional fractional diffusion equation. In this section, we study our proposed preconditioner for linear systems arising from a two-dimensional fractional diffusion equation.

3.1. Discretization matrices. Consider a two-dimensional initial-boundary value problem of space-fractional diffusion equation [9]:

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y,t)\frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} + e(x,y,t)\frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} + f(x,y,t),$$
$$(x,y,t) \in \Omega \times (0,T], \tag{17}$$
$$u(x,y,t) = 0, \qquad (x,y,t) \in \partial\Omega \times (0,T], \tag{18}$$
$$u(x,y,0) = \psi(x,y), \qquad (x,y) \in \bar{\Omega}, \tag{19}$$

where $\Omega = (x_L, x_R) \times (y_D, y_U)$, $\partial\Omega$ denotes the boundary of Ω, $d(x,y,t)$ and $e(x,y,t)$ are known functions that are larger than a positive constant, $f(x,y,t)$ is the source term, $\psi(x,y)$ is the initial condition, $\alpha, \beta \in (1,2)$, and the Riesz fractional derivatives are defined by [9]

$$\begin{aligned}\frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} &:= \sigma_\alpha\left({}_{x_L}D_x^\alpha + {}_xD_{x_R}^\alpha\right)u(x,y,t),\ (x,y,t) \in \Omega \times (0,T], \\ \frac{\partial^\beta u(x,y,t)}{\partial |y|^\beta} &:= \sigma_\beta\left({}_{y_D}D_y^\beta + {}_yD_{y_U}^\beta\right)u(x,y,t),\ (x,y,t) \in \Omega \times (0,T],\end{aligned}$$

with $\sigma_\alpha = -\frac{1}{2\cos(\frac{\pi\alpha}{2})} > 0$ and $\sigma_\beta = -\frac{1}{2\cos(\frac{\pi\beta}{2})} > 0$.

For positive integers M and N, let

$$\tau = T/N, \quad h_x = (x_R - x_L)/(M+1), \quad h_y = (y_U - y_D)/(M+1).$$

Define the temporal girds, the spatial grids in the x-direction, and the spatial grids in the y-direction by

$$\{t_n = n\tau | 0 \le n \le N\},\ \{x_i = x_L + ih_x | 0 \le i \le M+1\},\ \{y_j = y_D + jh_y | 0 \le j \le M+1\},$$

respectively. Then, the vectors consisting of spatial-grid points with x-dominant ordering and y-dominant ordering are defined, respectively, by

$$\mathscr{P}_{x,M} = (P_{1,1}, P_{2,1}, \ldots, P_{M,1}, P_{1,2}, P_{2,2}, \ldots, P_{M,2}, \ldots, P_{1,M}, P_{2,M}, \ldots, P_{M,M})^{\mathrm{T}}, \tag{20}$$

$$\mathscr{P}_{y,M} = (P_{1,1}, P_{1,2}, \ldots, P_{1,M}, P_{2,1}, P_{2,2}, \ldots, P_{2,M}, \ldots, P_{M,1}, P_{M,2}, \ldots, P_{M,M})^{\mathrm{T}}, \tag{21}$$

where $P_{i,j}$ denotes the point (x_i, y_j) for $0 \le i, j \le M+1$. Also, denote

$$d_{i,j,n} = d(x_i, y_j, t_n), \quad e_{i,j,n} = e(x_i, y_j, t_n), \quad 0 \le i, j \le M+1, \quad 0 \le n \le N.$$

Then, using (5) and the backward difference approximation to $\frac{\partial u}{\partial t}$, an implicit difference discretization of (17)–(19) is given as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} = -\left(\frac{1}{h_x^{\alpha}}\mathbf{D}_{n+1}\mathbf{B}_x + \frac{1}{h_y^{\beta}}\mathbf{E}_{n+1}\mathbf{B}_y\right)\mathbf{u}^{n+1} + \mathbf{f}^{n+1}, \quad 0 \le n \le N-1, \tag{22}$$

where $\mathbf{u}^n$ is an approximate solution to $u(\mathscr{P}_{x,M}, t_n)$ for $1 \le n \le N$, $\mathbf{u}^0 = \psi(\mathscr{P}_{x,M})$, $\mathbf{f}^n = f(\mathscr{P}_{x,M}, t_n)$ for $1 \le n \le N$, $\mathbf{D}_n = d(\mathscr{P}_{x,M}, t_n)$, $\mathbf{E}_n = e(\mathscr{P}_{x,M}, t_n)$, $\mathbf{B}_x = \mathbf{I}_M \otimes \mathbf{S}_\alpha$, $\mathbf{B}_y = \mathbf{S}_\beta \otimes \mathbf{I}_M$, "$\otimes$" denotes the Kronecker product, and $\mathbf{S}_\alpha$ and $\mathbf{S}_\beta$ are real symmetric positive definite Toeplitz matrices with their first columns given by $(s_0^{(\alpha)}, s_1^{(\alpha)}, \ldots, s_{M-1}^{(\alpha)})^{\mathrm{T}}$ and $(s_0^{(\beta)}, s_1^{(\beta)}, \ldots, s_{M-1}^{(\beta)})^{\mathrm{T}}$, respectively. The resulting task from (22) is to solve one after another the following N linear systems:

$$\mathbf{A}_n \mathbf{u}_n = \mathbf{b}_n, \qquad 1 \le n \le N, \tag{23}$$

where $\mathbf{A}_n = \mathbf{I}_{M^2} + \eta_x \mathbf{D}_n \mathbf{B}_x + \eta_y \mathbf{E}_n \mathbf{B}_y$, $\mathbf{b}_n = \tau \mathbf{f}^n + \mathbf{u}^{n-1}$, $\eta_x = \tau/h_x^{\alpha}$, and $\eta_y = \tau/h_y^{\beta}$. Notice that $\mathbf{A}_n$ is a block-Toeplitz-like matrix, matrix-vector multiplication of which can be doe fast with $\mathcal{O}(M^2 \log M)$ operation and $\mathcal{O}(M^2)$ storage using properties of the Kronecker product and FFTs; see [22].

The proposed splitting preconditioner $\mathbf{P}_n = \mathbf{W}_n \mathbf{T}_n$ can be developed for (23). Here,

$$\begin{aligned} \mathbf{T}_n &= \bar{\theta}_n \mathbf{I}_{M^2} + \eta_x \bar{d}_n \mathbf{B}_x + \eta_y \bar{e}_n \mathbf{B}_y, \quad \mathbf{W}_n = \mathbf{I}_{M^2} + \mathbf{D}_n + \mathbf{E}_n, \\ \bar{\theta}_n &= \mathtt{mean}(\mathbf{W}_n^{-1}), \quad \bar{d}_n = \mathtt{mean}(\mathbf{D}_n \mathbf{W}_n^{-1}), \quad \bar{e}_n = \mathtt{mean}(\mathbf{E}_n \mathbf{W}_n^{-1}). \end{aligned}$$

It is obvious that $\mathbf{P}_n$ is invertible for each n (similar to Proposition 2.1).

3.2. The singular values of preconditioned matrices. In this subsection, we study the spectral properties of the preconditioned matrices. Define functions

$$\mu(x,y,z) = \frac{xy^2(x_R - x_L)^{2-\alpha}}{2z(2-\alpha)},\ \omega(x,y,z) = \frac{xy^2(y_U - y_D)^{2-\beta}}{2z(2-\beta)},\ x, y \ge 0, \text{ and } z > 0. \tag{24}$$

THEOREM 3.1. *Assume*

(i) $d(x,y,t) \equiv d(x,t)$, $e(x,y,t) \equiv e(x,t)$ *and* $d(x,t), e(x,t) \in [\check{c},\hat{c}]$ *for any* $(x,y,t) \in \Omega \times (0,T]$ *with* $\check{c} > 0$,

(ii) *for any* $t \in (0,T]$, $d(\cdot,t), e(\cdot,t) \in \mathcal{L}((x_L,x_R))$ *with*

$$\sup_{t\in(0,T]} \max\left\{|d(\cdot,t)|_{\mathcal{L}((x_L,x_R))}, |e(\cdot,t)|_{\mathcal{L}((x_L,x_R))}\right\} \le \tilde{c}, \text{ and}$$

(iii) $\{s_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \le c_0$.

Then, for any $N \ge N_0$, $\bigcup_{n=1}^N \Sigma^2(\mathbf{A}_n\mathbf{P}_n^{-1}) \subset [\check{s},\hat{s}]$, *and thus*

$$\sup_{M\ge1}\sup_{N\ge N_0}\max_{1\le n\le N} \mathrm{cond}(\mathbf{A}_\mathrm{n}\mathbf{P}_\mathrm{n}^{-1}) \le \sqrt{\hat{\mathrm{s}}/\check{\mathrm{s}}},$$

where $\check{s}$, $\hat{s}$, *and* N_0 *are positive constants independent of* τ *and* h:

$$\check{s} = \frac{(1+2\check{c})^2\check{c}^2}{4(1+2\hat{c})^2\hat{c}^2},\ \hat{s} = \frac{1}{\check{s}},\ N_0 = 8T\max\{\mu(c_0,\hat{c}\tilde{c},\check{c}^3), 8(1+2\hat{c})^3\hat{c}\mu(c_0,\tilde{c},(1+2\check{c})^5)\}.$$

Proof. Denote $d_{i,n} = d(x_i,t_n)$ and $e_{i,n} = e(x_i,t_n)$ for $1 \le i \le M$. Since $d(x,y,t)$ and $e(x,y,t)$ are now independent of y, $\mathbf{D}_n$ and $\mathbf{E}_n$ can be rewritten as $\mathbf{D}_n = \mathbf{I}_M \otimes \mathbf{D}_x$ and $\mathbf{E}_n = \mathbf{I}_M \otimes \mathbf{E}_x$ with $\mathbf{D}_x = \mathrm{diag}(d_{1,n}, d_{2,n}, \ldots, d_{M,n})$ and $\mathbf{E}_x = \mathrm{diag}(e_{1,n}, e_{2,n}, \ldots, e_{M,n})$. Denote

$$\begin{aligned}
&\mathbf{M}_x = \mathbf{D}_x\mathbf{E}_x, \quad \mathbf{H}_1 = \mathbf{D}_x\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{D}_x, \quad \mathbf{H}_2 = \mathbf{M}_x\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{M}_x,\\
&\mathbf{W}_x = (\mathbf{I}_M + \mathbf{D}_x + \mathbf{E}_x)^2, \quad \mathbf{H}_3 = \mathbf{W}_x\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{W}_x, \quad \mathbf{Q}_{a1} = \mathbf{I}_{M^2} + \eta_x\mathbf{I}_M \otimes \mathbf{H}_1,\\
&\mathbf{Q}_{a2} = 2\eta_y\mathbf{S}_\beta \otimes \mathbf{E}_x + \eta_x\eta_y\mathbf{S}_\beta \otimes \mathbf{H}_2,\ \mathbf{Q}_{a3} = \eta_x^2\mathbf{I}_M \otimes (\mathbf{S}_\alpha\mathbf{D}_x^2\mathbf{S}_\alpha),\ \mathbf{Q}_{a4} = \eta_y^2\mathbf{S}_\beta^2 \otimes \mathbf{E}_x^2,\\
&\mathbf{Q}_{p1} = \bar{\theta}_n^2\mathbf{I}_M \otimes \mathbf{W}_x + \eta_x\bar{\theta}_n\bar{d}_n\mathbf{I}_M \otimes \mathbf{H}_3, \quad \mathbf{Q}_{p3} = \eta_x^2\bar{d}_n^2\mathbf{I}_M \otimes (\mathbf{S}_\alpha\mathbf{W}_x\mathbf{S}_\alpha),\\
&\mathbf{Q}_{p2} = 2\eta_y\bar{\theta}_n\bar{e}_n\mathbf{S}_\beta \otimes \mathbf{W}_x + \eta_x\eta_y\bar{d}_n\bar{e}_n\mathbf{S}_\beta \otimes \mathbf{H}_3, \quad \mathbf{Q}_{p4} = \eta_y^2\bar{e}_n^2\mathbf{S}_\beta^2 \otimes \mathbf{W}_x.
\end{aligned}$$

By straightforward calculation,

$$\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n = \mathbf{Q}_{a1} + \mathbf{Q}_{a2} + \mathbf{Q}_{a3} + \mathbf{Q}_{a4}, \tag{25}$$

$$\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n = \mathbf{Q}_{p1} + \mathbf{Q}_{p2} + \mathbf{Q}_{p3} + \mathbf{Q}_{p4}. \tag{26}$$

By assumption (i), $\mathtt{range}(\mathbf{D}_x) \subset [\check{c},\hat{c}]$. By assumption (ii),

$$\nabla(\mathbf{D}_x) = \max_{1\le i,j\le M, i\ne j} \frac{|d(x_i,t_n) - d(x_j,t_n)|}{|i-j|} \le \max_{1\le i,j\le M, i\ne j} \frac{\tilde{c}|x_i - x_j|}{|i-j|} = \tilde{c}h_x.$$

Hence, by Lemma 2.3, we have $\check{c}\mathbf{S}_\alpha - 2\mu(c_0,\tilde{c},\check{c})h_x^\alpha\mathbf{I}_M \prec \mathbf{H}_1 \prec 4\hat{c}\mathbf{S}_\alpha + \mu(c_0,\tilde{c},\hat{c})h_x^\alpha\mathbf{I}_M$. On the other hand, $N \ge N_0$ and $\hat{c} \ge \check{c} > 0$ imply that

$$\tau = \frac{T}{N} \le \frac{T}{N_0} = \max\left\{\frac{1}{8\mu(c_0,\hat{c}\tilde{c},\check{c}^3)}, \frac{1}{64(1+2\hat{c})^3\hat{c}\mu(c_0,\tilde{c},(1+2\check{c})^5)}\right\}. \tag{27}$$

Moreover, (27) and $\hat{c} \ge \check{c} > 0$ induce that

$$\tau \le \frac{1}{8\mu(c_0,\hat{c}\tilde{c},\check{c}^3)} \le \frac{1}{8\mu(c_0,\tilde{c},\check{c})} \le \frac{1}{8\mu(c_0,\tilde{c},\hat{c})}.$$

Therefore, $\eta_x\check{c}\mathbf{S}_\alpha - 4^{-1}\mathbf{I}_M \prec \eta_x\mathbf{H}_1 \prec 4\eta_x\hat{c}\mathbf{S}_\alpha + 8^{-1}\mathbf{I}_M$, and thus

$$(3/4)\mathbf{I}_{M^2} + \eta_x\check{c}\mathbf{I}_M \otimes \mathbf{S}_\alpha \prec \mathbf{Q}_{a1} \prec (9/8)\mathbf{I}_{M^2} + 4\eta_x\hat{c}\mathbf{I}_M \otimes \mathbf{S}_\alpha. \tag{28}$$

By assumption (i), $\texttt{range}(\mathbf{M}_x) \subset [\check{c}^2, \hat{c}^2]$. By assumptions (i) and (ii),

$$\begin{aligned}\nabla(\mathbf{M}_x) &= \max_{1\le i,j\le M, i\ne j} \frac{|d_{i,n}e_{i,n} - d_{j,n}e_{j,n}|}{|i-j|}\\ &= \max_{1\le i,j\le M, i\ne j} \frac{|d_{i,n}e_{i,n} - d_{i,n}e_{j,n} + d_{i,n}e_{j,n} - d_{j,n}e_{j,n}|}{|i-j|}\\ &\le \max_{1\le i,j\le M, i\ne j} \left(\frac{d_{i,n}|e_{i,n}-e_{j,n}|}{|i-j|} + \frac{e_{j,n}|d_{i,n}-d_{j,n}|}{|i-j|}\right)\\ &\le 2\hat{c}\tilde{c} \max_{1\le i,j\le M, i\ne j} \frac{|x_i - x_j|}{|i-j|} = 2\hat{c}\tilde{c}h_x.\end{aligned}$$

Hence, by Lemma 2.3, we have

$$\check{c}^2\mathbf{S}_\alpha - 2\mu(c_0, 2\hat{c}\tilde{c}, \check{c}^2)h_x^\alpha \mathbf{I}_M \prec \mathbf{H}_2 \prec 4\hat{c}^2\mathbf{S}_\alpha + \mu(c_0, 2\hat{c}\tilde{c}, \hat{c}^2)h_x^\alpha \mathbf{I}_M.$$

Moreover, (27) and $\hat{c} \ge \check{c} > 0$ induce that

$$\tau \le \frac{1}{8\mu(c_0, \hat{c}\tilde{c}, \check{c}^3)} = \frac{\check{c}}{2\mu(c_0, 2\hat{c}\tilde{c}, \check{c}^2)} \le \frac{\hat{c}}{2\mu(c_0, 2\hat{c}\tilde{c}, \hat{c}^2)}.$$

Hence, $\eta_x\check{c}^2\mathbf{S}_\alpha - \check{c}\mathbf{I}_M \prec \eta_x\mathbf{H}_2 \prec 4\eta_x\hat{c}^2\mathbf{S}_\alpha + 2^{-1}\hat{c}\mathbf{I}_M$. Moreover, assumption (i) induces that $\check{c}\mathbf{I}_M \preceq \mathbf{E}_x \preceq \hat{c}\mathbf{I}_M$. Therefore,

$$\eta_y\check{c}\mathbf{S}_\beta \otimes \mathbf{I}_M + \eta_x\eta_y\check{c}^2\mathbf{S}_\beta \otimes \mathbf{S}_\alpha \prec \mathbf{Q}_{a2} \prec (5/2)\eta_y\hat{c}\mathbf{S}_\beta \otimes \mathbf{I}_M + 4\eta_x\eta_y\hat{c}^2\mathbf{S}_\beta \otimes \mathbf{S}_\alpha. \tag{29}$$

By (25), (28), and (29),

$$\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n \succ \frac{3}{4}\mathbf{I}_{M^2} + \eta_x\check{c}\mathbf{I}_M \otimes \mathbf{S}_\alpha + \eta_y\check{c}\mathbf{S}_\beta \otimes \mathbf{I}_M + \eta_x\eta_y\check{c}^2\mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{a3} + \mathbf{Q}_{a4} \succ \mathbf{O}. \tag{30}$$

By (25), (28), and (29) again,

$$\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n \prec \frac{9}{8}\mathbf{I}_{M^2} + 4\eta_x\hat{c}\mathbf{I}_M \otimes \mathbf{S}_\alpha + \frac{5}{2}\eta_y\hat{c}\mathbf{S}_\beta \otimes \mathbf{I}_M + 4\eta_x\eta_y\hat{c}^2\mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{a3} + \mathbf{Q}_{a4}. \tag{31}$$

Denote $a_w = 1 + 2\check{c}$ and $b_w = 1 + 2\hat{c}$. By assumption (i), we obtain $\texttt{range}(\mathbf{W}_x) \subset [a_w^2, b_w^2]$. By assumptions (i) and (ii),

$$\begin{aligned}\nabla(\mathbf{W}_x) &= \max_{1\le i,j\le M, i\ne j} \frac{|(1+d_{i,n}+e_{i,n})^2 - (1+d_{j,n}+e_{j,n})^2|}{|i-j|}\\ &= \max_{1\le i,j\le M, i\ne j} \frac{|1+d_{i,n}+e_{i,n}+1+d_{j,n}+e_{j,n}||d_{i,n}-d_{j,n}+e_{i,n}-e_{j,n}|}{|i-j|}\\ &\le 4b_w\tilde{c} \max_{1\le i,j\le M, i\ne j} \frac{|x_i - x_j|}{|i-j|} = 4b_w\tilde{c}h_x.\end{aligned}$$

Hence, by Lemma 2.3, we get

$$a_w^2\mathbf{S}_\alpha - 2\mu(c_0, 4b_w\tilde{c}, a_w^2)h_x^\alpha\mathbf{I}_M \prec \mathbf{H}_3 \prec 4b_w^2\mathbf{S}_\alpha + \mu(c_0, 4b_w\tilde{c}, b_w^2)h_x^\alpha\mathbf{I}_M.$$

Moreover, (27) and $\hat{c} \ge \check{c} > 0$ induce that

$$\tau \le \frac{1}{64(1+2\hat{c})^3\hat{c}\mu(c_0, \tilde{c}, (1+2\check{c})^5)} = \frac{a_w^3}{4b_w\hat{c}\mu(c_0, 4b_w\tilde{c}, a_w^2)} \le \frac{b_w^2}{4\hat{c}\mu(c_0, 4b_w\tilde{c}, b_w^2)}.$$

Hence,

$$\eta_x a_w^2 \mathbf{S}_\alpha - 2^{-1} b_w^{-1} \hat{c}^{-1} a_w^3 \mathbf{I}_M \prec \eta_x \mathbf{H}_3 \prec 4\eta_x b_w^2 \mathbf{S}_\alpha + 4^{-1} \hat{c}^{-1} b_w^2 \mathbf{I}_M, \tag{32}$$

which implies that

$$\begin{cases} \mathbf{Q}_{p1} \succ \bar{\theta}_n(\bar{\theta}_n \mathbf{I}_M \otimes \mathbf{W}_x - 2^{-1} b_w^{-1} \hat{c}^{-1} a_w^3 \bar{d}_n \mathbf{I}_{M^2}) + \eta_x \bar{\theta}_n \bar{d}_n a_w^2 \mathbf{I}_M \otimes \mathbf{S}_\alpha, \\ \mathbf{Q}_{p1} \prec \bar{\theta}_n(\bar{\theta}_n \mathbf{I}_M \otimes \mathbf{W}_x + 4^{-1} \hat{c}^{-1} b_w^2 \bar{d}_n \mathbf{I}_{M^2}) + 4\eta_x \bar{\theta}_n \bar{d}_n b_w^2 \mathbf{I}_M \otimes \mathbf{S}_\alpha. \end{cases}$$

Moreover, assumption (i) implies that

$$b_w^{-1} \le \bar{\theta}_n \le a_w^{-1}, \quad b_w^{-1} \check{c} \le \bar{d}_n, \bar{e}_n \le a_w^{-1} \hat{c}, \quad a_w^2 \mathbf{I}_M \preceq \mathbf{W}_x \preceq b_w^2 \mathbf{I}_M. \tag{33}$$

Hence, we have

$$2^{-1} b_w^{-2} a_w^2 \mathbf{I}_{M^2} + \eta_x b_w^{-2} a_w^2 \check{c} \mathbf{I}_M \otimes \mathbf{S}_\alpha \prec \mathbf{Q}_{p1} \prec (5/4) a_w^{-2} b_w^2 \mathbf{I}_{M^2} + 4\eta_x a_w^{-2} b_w^2 \hat{c} \mathbf{I}_M \otimes \mathbf{S}_\alpha. \tag{34}$$

Equation (32) also implies that

$$\begin{cases} \mathbf{Q}_{p2} \succ \eta_y \bar{e}_n [\mathbf{S}_\beta \otimes (2\bar{\theta}_n \mathbf{W}_x - 2^{-1} b_w^{-1} \hat{c}^{-1} a_w^3 \bar{d}_n \mathbf{I}_M + \eta_x a_w^2 \bar{d}_n \mathbf{S}_\alpha)], \\ \mathbf{Q}_{p2} \prec \eta_y \bar{e}_n [\mathbf{S}_\beta \otimes (2\bar{\theta}_n \mathbf{W}_x + 4^{-1} \hat{c}^{-1} b_w^2 \bar{d}_n \mathbf{I}_M + 4\eta_x b_w^2 \bar{d}_n \mathbf{S}_\alpha)], \end{cases}$$

which together with (33) yields that

$$\frac{3\eta_y a_w^2 \check{c}}{2b_w^2} \mathbf{S}_\beta \otimes \mathbf{I}_M + \frac{\eta_x \eta_y a_w^2 \check{c}^2}{b_w^2} \mathbf{S}_\beta \otimes \mathbf{S}_\alpha \prec \mathbf{Q}_{p2} \prec \frac{9\eta_y b_w^2 \hat{c}}{4a_w^2} \mathbf{S}_\beta \otimes \mathbf{I}_M + \frac{4\eta_x \eta_y b_w^2 \hat{c}^2}{a_w^2} \mathbf{S}_\beta \otimes \mathbf{S}_\alpha. \tag{35}$$

By (26), (34), and (35),

$$\mathbf{P}_n^{\mathrm{T}} \mathbf{P}_n \succeq \frac{a_w^2}{2b_w^2} \mathbf{I}_{M^2} + \frac{\eta_x a_w^2 \check{c}}{b_w^2} \mathbf{I}_M \otimes \mathbf{S}_\alpha + \frac{3\eta_y a_w^2 \check{c}}{2b_w^2} \mathbf{S}_\beta \otimes \mathbf{I}_M + \frac{\eta_x \eta_y a_w^2 \check{c}^2}{b_w^2} \mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{p3} + \mathbf{Q}_{p4}. \tag{36}$$

By (26), (34), and (35) again,

$$\mathbf{P}_n^{\mathrm{T}} \mathbf{P}_n \preceq \frac{5b_w^2}{4a_w^2} \mathbf{I}_{M^2} + \frac{4\eta_x b_w^2 \hat{c}}{a_w^2} \mathbf{I}_M \otimes \mathbf{S}_\alpha + \frac{9\eta_y b_w^2 \hat{c}}{4a_w^2} \mathbf{S}_\beta \otimes \mathbf{I}_M + \frac{4\eta_x \eta_y b_w^2 \hat{c}^2}{a_w^2} \mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{p3} + \mathbf{Q}_{p4}. \tag{37}$$

For any nonzero vector $\mathbf{y} \in \mathbb{R}^{M^2 \times 1}$, denote $\mathbf{z} = \mathbf{P}_n^{-1} \mathbf{y}$. Then, it holds that

$$\frac{\mathbf{y}^{\mathrm{T}} (\mathbf{A}_n \mathbf{P}_n^{-1})^{\mathrm{T}} (\mathbf{A}_n \mathbf{P}_n^{-1}) \mathbf{y}}{\mathbf{y}^{\mathrm{T}} \mathbf{y}} = \frac{\mathbf{z}^{\mathrm{T}} \mathbf{A}_n^{\mathrm{T}} \mathbf{A}_n \mathbf{z}}{\mathbf{z}^{\mathrm{T}} \mathbf{P}_n^{\mathrm{T}} \mathbf{P}_n \mathbf{z}}.$$

Denote $\sigma = \frac{\mathbf{z}^{\mathrm{T}} \mathbf{A}_n^{\mathrm{T}} \mathbf{A}_n \mathbf{z}}{\mathbf{z}^{\mathrm{T}} \mathbf{P}_n^{\mathrm{T}} \mathbf{P}_n \mathbf{z}}$. Notice that matrices involved on the right-hand sides of (30), (31), (36), and (37) are all real symmetric positive definite. Moreover, by (i) and (33), it is easy to check that

$$\frac{a_w^2 \check{c}^2}{b_w^2 \hat{c}^2} \le \frac{\mathbf{z}^{\mathrm{T}} \mathbf{Q}_{a3} \mathbf{z}}{\mathbf{z}^{\mathrm{T}} \mathbf{Q}_{p3} \mathbf{z}} \le \frac{b_w^2 \hat{c}^2}{a_w^2 \check{c}^2}, \qquad \frac{a_w^2 \check{c}^2}{b_w^2 \hat{c}^2} \le \frac{\mathbf{z}^{\mathrm{T}} \mathbf{Q}_{a4} \mathbf{z}}{\mathbf{z}^{\mathrm{T}} \mathbf{Q}_{p4} \mathbf{z}} \le \frac{b_w^2 \hat{c}^2}{a_w^2 \check{c}^2}. \tag{38}$$

Hence, the application of Proposition 2.4 to (31), (36), and (38) yields

$$\begin{aligned} \sigma &\le \frac{\mathbf{z}^{\mathrm{T}} \left[\frac{9}{8} \mathbf{I}_{M^2} + 4\eta_x \hat{c} \mathbf{I}_M \otimes \mathbf{S}_\alpha + \frac{5}{2} \eta_y \hat{c} \mathbf{S}_\beta \otimes \mathbf{I}_M + 4\eta_x \eta_y \hat{c}^2 \mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{a3} + \mathbf{Q}_{a4} \right] \mathbf{z}}{\mathbf{z}^{\mathrm{T}} \left[\frac{a_w^2}{2b_w^2} \mathbf{I}_{M^2} + \frac{\eta_x a_w^2 \check{c}}{b_w^2} \mathbf{I}_M \otimes \mathbf{S}_\alpha + \frac{3\eta_y a_w^2 \check{c}}{2b_w^2} \mathbf{S}_\beta \otimes \mathbf{I}_M + \frac{\eta_x \eta_y a_w^2 \check{c}^2}{b_w^2} \mathbf{S}_\beta \otimes \mathbf{S}_\alpha + \mathbf{Q}_{p3} + \mathbf{Q}_{p4} \right] \mathbf{z}} \\ &\le \max \left\{ \frac{9b_w^2}{4a_w^2}, \frac{4b_w^2 \hat{c}}{a_w^2 \check{c}}, \frac{5b_w^2 \hat{c}}{3a_w^2 \check{c}}, \frac{4b_w^2 \hat{c}^2}{a_w^2 \check{c}^2}, \frac{b_w^2 \hat{c}^2}{a_w^2 \check{c}^2} \right\} = \hat{s}. \end{aligned} \tag{39}$$

Applying Proposition 2.4 to (30), (37), and (38) again, we get

$$\sigma \geq \frac{\mathbf{z}^{\mathrm{T}}\left[\frac{3}{4}\mathbf{I}_{M^2}+\eta_x\check{c}\mathbf{I}_M\otimes\mathbf{S}_\alpha+\eta_y\check{c}\mathbf{S}_\beta\otimes\mathbf{I}_M+\eta_x\eta_y\check{c}^2\mathbf{S}_\beta\otimes\mathbf{S}_\alpha+\mathbf{Q}_{a3}+\mathbf{Q}_{a4}\right]\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\left[\frac{5b_w^2}{4a_w^2}\mathbf{I}_{M^2}+\frac{4\eta_x b_w^2\hat{c}}{a_w^2}\mathbf{I}_M\otimes\mathbf{S}_\alpha+\frac{9\eta_y b_w^2\hat{c}}{4a_w^2}\mathbf{S}_\beta\otimes\mathbf{I}_M+\frac{4\eta_x\eta_y b_w^2\hat{c}^2}{a_w^2}\mathbf{S}_\beta\otimes\mathbf{S}_\alpha+\mathbf{Q}_{p3}+\mathbf{Q}_{p4}\right]\mathbf{z}}$$

$$\geq \min\left\{\frac{3a_w^2}{5b_w^2},\frac{a_w^2\check{c}}{4b_w^2\hat{c}},\frac{4a_w^2\check{c}}{9b_w^2\hat{c}},\frac{a_w^2\check{c}^2}{4b_w^2\hat{c}^2},\frac{a_w^2\check{c}^2}{b_w^2\hat{c}^2}\right\}=\check{s}. \tag{40}$$

The result follows from (39) and (40). ∎

Similarly, we can deal with the case $d(x,y,t)\equiv d(y,t)$ and $e(x,y,t)\equiv e(y,t)$. Define a permutation matrix $\widehat{\mathbf{P}}$ such that

$$\mathscr{P}_{y,M}=\widehat{\mathbf{P}}\mathscr{P}_{x,M}, \tag{41}$$

where $\mathscr{P}_{x,M}$ and $\mathscr{P}_{y,M}$ are two vectors defined in (20)–(21).

Theorem 3.2. *Assume*

(i) $d(x,y,t), e(x,y,t)\in[\check{c},\hat{c}]$ *and* $d(x,y,t)\equiv d(y,t)$, $e(x,y,t)\equiv e(y,t)$ *for any* $(x,y,t)\in\Omega\times(0,T]$ *with* $\check{c}>0$,

(ii) *for any* $t\in(0,T]$, $d(\cdot,t),e(\cdot,t)\in\mathcal{L}((y_D,y_U))$ *with*

$$\sup_{t\in(0,T]}\max\left\{|d(\cdot,t)|_{\mathcal{L}((y_D,y_U))},|e(\cdot,t)|_{\mathcal{L}((y_D,y_U))}\right\}\leq\tilde{c},\ \textit{and}$$

(iii) $\{s_k^{(\beta)}\}_{k\geq0}\in\mathcal{D}_\beta$ *with* $||\{s_k^{(\beta)}\}||_{\mathcal{D}_\beta}\leq c_0$.

Then, for any $N\geq N_0$, $\bigcup_{n=1}^N\Sigma^2(\mathbf{A}_n\mathbf{P}_n^{-1})\subset[\check{s},\hat{s}]$, *and thus*

$$\sup_{M\geq1}\sup_{N\geq N_0}\max_{1\leq n\leq N}\mathrm{cond}(\mathbf{A}_\mathrm{n}\mathbf{P}_\mathrm{n}^{-1})\leq\sqrt{\hat{\mathrm{s}}/\check{\mathrm{s}}},$$

where $\check{s}$, $\hat{s}$, *and* N_0 *are positive constants independent of* τ *and* h*:*

$$\check{s}=\frac{(1+2\check{c})^2\check{c}^2}{4(1+2\hat{c})^2\hat{c}^2},\ \hat{s}=\frac{1}{\check{s}},\ N_0=8T\max\{\omega(c_0,\hat{c}\tilde{c},\check{c}^3),8(1+2\hat{c})^2\hat{c}\omega(c_0,\tilde{c},(1+2\check{c})^5)\}.$$

Proof. Since $d(x,y,t)$ and $e(x,y,t)$ are now independent of x, $\mathbf{D}_n$ and $\mathbf{E}_n$ can be rewritten as $\mathbf{D}_n=\mathbf{D}_y\otimes\mathbf{I}_M$, $\mathbf{E}_n=\mathbf{E}_y\otimes\mathbf{I}_M$ with

$$\mathbf{D}_y=\mathrm{diag}(d(y_1,t_n),d(y_2,t_n),\ldots,d(y_M,t_n)),$$
$$\mathbf{E}_y=\mathrm{diag}(e(y_1,t_n),e(y_2,t_n),\ldots,e(y_M,t_n)).$$

Notice that $\widehat{\mathbf{P}}^{\mathrm{T}}\mathbf{A}_n\mathbf{P}_n^{-1}\widehat{\mathbf{P}}$ and $\mathbf{A}_n\mathbf{P}_n^{-1}$ have the same set of singular values. Thus, it suffices to show that

$$\bigcup_{n=1}^N\Sigma^2((\widehat{\mathbf{P}}^{\mathrm{T}}\mathbf{A}_n\widehat{\mathbf{P}})(\widehat{\mathbf{P}}^{\mathrm{T}}\mathbf{P}_n\widehat{\mathbf{P}})^{-1})\subset[\check{s},\hat{s}]\quad\forall N\geq N_0. \tag{42}$$

We find that $\widehat{\mathbf{P}}^{\mathrm{T}}\mathbf{A}_n\widehat{\mathbf{P}}=\mathbf{I}_{M^2}+\eta_y(\mathbf{I}_M\otimes\mathbf{E}_y)(\mathbf{I}_M\otimes\mathbf{S}_\beta)+\eta_x(\mathbf{I}_M\otimes\mathbf{D}_y)(\mathbf{S}_\alpha\otimes\mathbf{I}_M)$ and $\widehat{\mathbf{P}}^{\mathrm{T}}\mathbf{P}_n\widehat{\mathbf{P}}=[\mathbf{I}_M\otimes(\mathbf{I}_M+\mathbf{D}_y+\mathbf{E}_y)](\bar{\theta}_n\mathbf{I}_{M^2}+\eta_y\bar{e}_n\mathbf{I}_M\otimes\mathbf{S}_\beta+\eta_x\bar{d}_n\mathbf{S}_\alpha\otimes\mathbf{I}_M)$. Hence, the proof of (42) is similar to that of Theorem 3.1. ∎

In Theorems 3.1 and 3.2, $d(x,y,t)$ and $e(x,y,t)$ are assumed to be x-independent or y-independent. In the next theorem, we consider the case $d(x,y,t)=\nu_1(t)a(x,y,t)$

and $e(x,y,t) = \nu_2(t)a(x,y,t)$ for some nonnegative functions $\nu_1(t)$, $\nu_2(t)$ and positive function $a(x,y,t)$. For a real diagonal matrix,

(43)

$$\mathbf{V} = \mathrm{diag}(v_{1,1}, v_{2,1}, \dots, v_{M,1}, v_{1,2}, v_{2,2}, \dots, v_{M,2}, \dots, v_{1,M}, v_{2,M}, \dots, v_{M,M}) \in \mathbb{R}^{M^2\times M^2}$$

denotes

$$\nabla_1(\mathbf{V}) := \max_{1\le k\le M} \max_{1\le i,j\le M, i\ne j} \frac{|v_{i,k} - v_{j,k}|}{|i-j|}, \qquad \nabla_2(\mathbf{V}) := \max_{1\le k\le M} \max_{1\le i,j\le M, i\ne j} \frac{|v_{k,i} - v_{k,j}|}{|i-j|}.$$

Now we establish the following two lemmas for $\mathbf{V}$.

LEMMA 3.3. *Let* $\mathbf{V}$ *be of form* (43). *Assume*

(i) $\mathtt{range}(\mathbf{V}) \subset [\check{c}, \hat{c}]$ *with* $\check{c} > 0$,

(ii) $\nabla_1(\mathbf{V}) \le \tilde{c} h_x$, *and*

(iii) $\{s_k^{(\alpha)}\}_{k\ge 0} \in \mathcal{D}_\alpha$ *with* $||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha} \le c_0$.

Then, $\check{c}\mathbf{B}_x - 2\mu(c_0, \tilde{c}, \check{c})h_x^\alpha \mathbf{I}_{M^2} \prec \mathbf{V}\mathbf{B}_x + \mathbf{B}_x\mathbf{V} \prec 4\hat{c}\mathbf{B}_x + \mu(c_0, \tilde{c}, \hat{c})h_x^\alpha \mathbf{I}_{M^2}$, *where the function* $\mu(\cdot,\cdot,\cdot)$ *is defined in* (24).

Proof. Rewrite $\mathbf{B}_x$ and $\mathbf{V}$ as

$$\mathbf{B}_x = \mathbf{I}_M \otimes \mathbf{S}_\alpha, \quad \mathbf{V} = \mathrm{diag}(\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_M), \quad \mathbf{V}_i \in \mathbb{R}^{M\times M}, \ 1 \le i \le M.$$

Then, $\mathbf{V}\mathbf{B}_x + \mathbf{B}_x\mathbf{V}$ can be rewritten as

$$\mathbf{V}\mathbf{B}_x + \mathbf{B}_x\mathbf{V} = \mathrm{diag}(\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2, \dots, \tilde{\mathbf{H}}_M), \quad \tilde{\mathbf{H}}_i = \mathbf{V}_i\mathbf{S}_\alpha + \mathbf{S}_\alpha\mathbf{V}_i, \quad 1 \le i \le M.$$

By assumption (i), $\bigcup_{i=1}^M \mathtt{range}(\mathbf{V}_i) \subset [\check{c}, \hat{c}]$. Also, by assumption (ii), we get $\max_{1\le i\le M} \nabla(\mathbf{V}_i) = \nabla_1(\mathbf{V}) \le \tilde{c}h_x$. Hence, by Lemma 2.3, $\check{c}\mathbf{S}_\alpha - 2\mu(c_0, \tilde{c}, \check{c})h_x^\alpha \mathbf{I}_M \prec \tilde{\mathbf{H}}_i \prec 4\hat{c}\mathbf{S}_\alpha + \mu(c_0, \tilde{c}, \hat{c})h_x^\alpha \mathbf{I}_M$, for $1 \le i \le M$, which implies $\check{c}\mathbf{B}_x - 2\mu(c_0, \tilde{c}, \check{c})h_x^\alpha \mathbf{I}_{M^2} \prec \mathrm{diag}(\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2, \dots, \tilde{\mathbf{H}}_M) \prec 4\hat{c}\mathbf{B}_x + \mu(c_0, \tilde{c}, \hat{c})h_x^\alpha \mathbf{I}_{M^2}$. The result follows. ∎

By using an argument similar to Lemma 3.3 and applying the permutation matrix $\widehat{\mathbf{P}}$ in a way similar to the proof of Theorem 3.2, one can easily prove the following lemma.

LEMMA 3.4. *Let* $\mathbf{V}$ *be of form* (43). *Assume*

(i) $\mathtt{range}(\mathbf{V}) \subset [\check{c}, \hat{c}]$ *with* $\check{c} > 0$,

(ii) $\nabla_2(\mathbf{V}) \le \tilde{c} h_y$, *and*

(iii) $\{s_k^{(\beta)}\}_{k\ge 0} \in \mathcal{D}_\beta$ *with* $||\{s_k^{(\beta)}\}||_{\mathcal{D}_\beta} \le c_0$.

Then, $\check{c}\mathbf{B}_y - 2\omega(c_0, \tilde{c}, \check{c})h_y^\beta \mathbf{I}_{M^2} \prec \mathbf{V}\mathbf{B}_y + \mathbf{B}_y\mathbf{V} \prec 4\hat{c}\mathbf{B}_x + \omega(c_0, \tilde{c}, \hat{c})h_y^\beta \mathbf{I}_{M^2}$, *where the function* $\omega(\cdot,\cdot,\cdot)$ *is defined in* (24).

THEOREM 3.5. *Let* $d(x,y,t) \equiv \nu_1(t)a(x,y,t)$ *and* $e(x,y,t) \equiv \nu_2(t)a(x,y,t)$ *for any* $(x,y,t) \in \Omega \times (0,T]$. *Assume*

(i) $a(x,y,t) \in [\check{c}, \hat{c}]$ *with* $\check{c} > 0$ *for any* $(x,y,t) \in \Omega \times (0,T]$ *and* $\nu_1(t), \nu_2(t) \in [0, \hat{\nu}]$ *with* $\hat{\nu} > 0$ *for any* $t \in (0,T]$,

(ii) $a(\cdot,y,t) \in \mathcal{L}((x_L,x_R))$ *and* $a(x,\cdot,t) \in \mathcal{L}((y_D,y_U))$ *for any* $(x,y,t) \in \Omega \times (0,T]$ *with*

$$\max\left\{ \sup_{(y,t)\in\Omega_{y,t}} |a(\cdot,y,t)|_{\mathcal{L}((x_L,x_R))}, \sup_{(x,t)\in\Omega_{x,t}} |a(x,\cdot,t)|_{\mathcal{L}((y_D,y_U))} \right\} \le \tilde{c},$$

where $\Omega_{y,t} = (y_D, y_U) \times (0,T]$ *and* $\Omega_{x,t} = (x_L, x_R) \times (0,T]$, *and*

(iii) $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *and* $\{s_k^{(\beta)}\}_{k\geq 0} \in \mathcal{D}_\beta$ *with* $\max\{||\{s_k^{(\alpha)}\}||_{\mathcal{D}_\alpha}, ||\{s_k^{(\beta)}\}||_{\mathcal{D}_\beta}\} \leq c_0$.

Then, for any $N \geq N_0$, $\bigcup_{n=1}^{N} \Sigma^2(\mathbf{A}_n\mathbf{P}_n^{-1}) \subset [\check{s}, \hat{s}]$, *and thus*

$$\sup_{M\geq 1}\sup_{N\geq N_0}\max_{1\leq n\leq N} \mathrm{cond}(\mathbf{A}_\mathrm{n}\mathbf{P}_\mathrm{n}^{-1}) \leq \sqrt{\hat{\mathrm{s}}/\check{\mathrm{s}}},$$

where $\check{s}$, $\hat{s}$, *and* N_0 *are positive constants independent of* τ *and* h*:*

$$\check{s} = \min\left\{\frac{\check{c}}{4\hat{c}\hat{w}^2}, \frac{\breve{a}^2}{\hat{c}^2}\right\}, \quad \hat{s} = \max\left\{\frac{4\hat{c}\hat{w}}{\breve{a}}, \frac{\hat{c}^2}{\breve{a}^2}\right\}, \quad \hat{w} = 1 + 2\hat{v}\hat{c}, \quad \breve{a} = \check{c}/(1+2\hat{\nu}\check{c}),$$
$$N_0 = 8T\hat{\nu}\max\{\mu(c_0,\tilde{c},\check{c}), \omega(c_0,\tilde{c},\check{c}), \hat{w}\hat{c}\mu(c_0,\tilde{w},1), \hat{w}\hat{c}\omega(c_0,\tilde{w},1)\}, \quad \tilde{w} = 4\hat{w}\hat{v}\tilde{c},$$

where the functions $\mu(\cdot,\cdot,\cdot)$, $\omega(\cdot,\cdot,\cdot)$ *are defined in* (24).

Proof. Denote

$$\mathbf{D}_a = \mathrm{diag}(a_{1,1,n}, a_{2,1,n}, \ldots, a_{M,1,n}, a_{1,2,n}, a_{2,2,n}, \ldots, a_{M,2,n}, \ldots, a_{1,M,n}, \ldots, a_{M,M,n}),$$
$$a_{i,j,n} = a(x_i, y_j, t_n), \quad \bar{a} = \mathtt{mean}(\mathbf{D}_a\mathbf{W}_n^{-1}), \quad \nu_{1,n} = \nu_1(t_n), \quad \nu_{2,n} = \nu_2(t_n).$$

Since $d(x,y,t) \equiv \nu_1(t)a(x,y,t)$ and $e(x,y,t) \equiv \nu_2(t)a(x,y,t)$, it is easy to check that

$$\bar{d}_n = \nu_{1,n}\bar{a}, \quad \bar{e}_n = \nu_{2,n}\bar{a}, \quad \mathbf{D}_n = \nu_{1,n}\mathbf{D}_a, \quad \mathbf{E}_n = \nu_{2,n}\mathbf{D}_a.$$

Denote

$$\tilde{\mathbf{H}}_1 = \mathbf{B}_x\mathbf{D}_a + \mathbf{D}_a\mathbf{B}_x, \quad \tilde{\mathbf{H}}_2 = \mathbf{B}_y\mathbf{D}_a + \mathbf{D}_a\mathbf{B}_y, \quad \tilde{\mathbf{Q}}_{a1} = \mathbf{I}_{M^2} + \eta_x\nu_{1,n}\tilde{\mathbf{H}}_1 + \eta_y\nu_{2,n}\tilde{\mathbf{H}}_2,$$
$$\mathbf{B} = \eta_x\nu_{1,n}\mathbf{B}_x + \eta_y\nu_{2,n}\mathbf{B}_y, \quad \tilde{\mathbf{Q}}_{a2} = \mathbf{B}\mathbf{D}_a^2\mathbf{B}, \quad \tilde{\mathbf{H}}_3 = \mathbf{B}_x\mathbf{W}_n^2 + \mathbf{W}_n^2\mathbf{B}_x,$$
$$\tilde{\mathbf{H}}_4 = \mathbf{B}_y\mathbf{W}_n^2 + \mathbf{W}_n^2\mathbf{B}_y, \quad \tilde{\mathbf{Q}}_{p1} = \bar{\theta}_n^2\mathbf{W}_n^2 + \eta_x\bar{\theta}_n\bar{a}\nu_{1,n}\tilde{\mathbf{H}}_3 + \eta_y\bar{\theta}_n\bar{a}\nu_{2,n}\tilde{\mathbf{H}}_4,$$
$$\tilde{\mathbf{Q}}_{p2} = \bar{a}^2\mathbf{B}\mathbf{W}_n^2\mathbf{B}.$$

By straightforward calculation,

$$\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n = \tilde{\mathbf{Q}}_{a1} + \tilde{\mathbf{Q}}_{a2}, \tag{44}$$
$$\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n = \tilde{\mathbf{Q}}_{p1} + \tilde{\mathbf{Q}}_{p2}. \tag{45}$$

By assumptions (i) and (ii), we have $\mathtt{range}(\mathbf{D}_a) \subset [\check{c}, \hat{c}]$ and

$$\nabla_1(\mathbf{D}_a) = \max_{1\leq k\leq M}\max_{1\leq i,j\leq M, i\neq j}\frac{|a_{i,k,n} - a_{j,k,n}|}{|i-j|} \leq \tilde{c}\max_{1\leq i,j\leq M, i\neq j}\frac{|x_i - x_j|}{|i-j|} = \tilde{c}h_x.$$

Hence, by Lemma 3.3, we obtain

$$\check{c}\mathbf{B}_x - 2\mu(c_0,\tilde{c},\check{c})h_x^\alpha\mathbf{I}_{M^2} \prec \tilde{\mathbf{H}}_1 \prec 4\hat{c}\mathbf{B}_x + \mu(c_0,\tilde{c},\hat{c})h_x^\alpha\mathbf{I}_{M^2}.$$

By $N \geq N_0$,

$$\tau = \frac{T}{N} \leq \frac{T}{N_0} = \frac{1}{8\hat{\nu}}\min\left\{\frac{1}{\mu(c_0,\tilde{c},\check{c})}, \frac{1}{\omega(c_0,\tilde{c},\check{c})}, \frac{1}{\hat{w}\hat{c}\mu(c_0,\tilde{w},1)}, \frac{1}{\hat{w}\hat{c}\omega(c_0,\tilde{w},1)}\right\}. \tag{46}$$

Equation (46) and $\hat{c} \geq \check{c} > 0$ further imply that

$$\tau \leq \frac{1}{8\hat{\nu}\mu(c_0,\tilde{c},\check{c})} \leq \frac{1}{8\hat{\nu}\mu(c_0,\tilde{c},\hat{c})}.$$

Hence, $\eta_x\check{c}\mathbf{B}_x-(4\hat{\nu})^{-1}\mathbf{I}_{M^2}\prec\eta_x\tilde{\mathbf{H}}_1\prec 4\eta_x\hat{c}\mathbf{B}_x+(8\hat{\nu})^{-1}\mathbf{I}_{M^2}$, which together with the fact that $\nu_{1,n}\in[0,\hat{\nu}]$ implies that

$$\eta_x\nu_{1,n}\check{c}\mathbf{B}_x-4^{-1}\mathbf{I}_{M^2}\prec\eta_x\nu_{1,n}\tilde{\mathbf{H}}_1\prec 4\eta_x\nu_{1,n}\hat{c}\mathbf{B}_x+8^{-1}\mathbf{I}_{M^2}. \tag{47}$$

By assumption (ii), we have

$$\nabla_2(\mathbf{D}_a)=\max_{1\le k\le M}\max_{1\le i,j\le M,i\neq j}\frac{|a_{k,i,n}-a_{k,j,n}|}{|i-j|}\le\tilde{c}\max_{1\le i,j\le M,i\neq j}\frac{|y_i-y_j|}{|i-j|}=\tilde{c}h_y.$$

Applying Lemma 3.4 to $\texttt{range}(\mathbf{D}_a)\subset[\check{c},\hat{c}]$, $\nabla_2(\mathbf{D}_n)\le\tilde{c}h_y$, and assumption (iii) yields

$$\check{c}\mathbf{B}_y-2\omega(c_0,\tilde{c},\check{c})h_y^{\beta}\mathbf{I}_{M^2}\prec\tilde{\mathbf{H}}_2\prec 4\hat{c}\mathbf{B}_y+\omega(c_0,\tilde{c},\hat{c})h_y^{\beta}\mathbf{I}_{M^2}.$$

Equation (46) and $\hat{c}\ge\check{c}>0$ imply that

$$\tau\le\frac{1}{8\hat{\nu}\omega(c_0,\tilde{c},\check{c})}\le\frac{1}{8\hat{\nu}\omega(c_0,\tilde{c},\hat{c})}.$$

Moreover, $\nu_{2,n}\in[0,\hat{\nu}]$. Hence, we get

$$\eta_y\nu_{2,n}\check{c}\mathbf{B}_y-4^{-1}\mathbf{I}_{M^2}\prec\eta_y\nu_{2,n}\tilde{\mathbf{H}}_2\prec 4\eta_y\nu_{2,n}\hat{c}\mathbf{B}_y+8^{-1}\mathbf{I}_{M^2}, \tag{48}$$

which together with (47) implies that $2^{-1}\mathbf{I}_{M^2}+\check{c}\mathbf{B}\prec\tilde{\mathbf{Q}}_{a1}\prec(5/4)\mathbf{I}_{M^2}+4\hat{c}\mathbf{B}$. Hence, from (44), we see that

$$\mathbf{O}\prec 2^{-1}\mathbf{I}_{M^2}+\check{c}\mathbf{B}+\tilde{\mathbf{Q}}_{a2}\prec\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\prec(5/4)\mathbf{I}_{M^2}+4\hat{c}\mathbf{B}+\tilde{\mathbf{Q}}_{a2}. \tag{49}$$

By assumption (i),

$$\texttt{range}(\mathbf{W}_n^2)=\texttt{range}((1+\nu_{1,n}\mathbf{D}_a+\nu_{2,n}\mathbf{D}_a)^2)\subset[1,\hat{w}^2]. \tag{50}$$

By assumption (ii),

$$\begin{aligned}\nabla_1(\mathbf{W}_n^2)&=\max_{1\le k\le M}\max_{1\le i,j\le M,i\neq j}\frac{|[1+(\nu_{1,n}+\nu_{2,n})a_{i,k,n}]^2-[1+(\nu_{1,n}+\nu_{2,n})a_{j,k,n}]^2|}{|i-j|}\\&\le\max_{1\le k\le M}\max_{1\le i,j\le M,i\neq j}\frac{|2+2\hat{\nu}(a_{i,k,n}+a_{j,k,n})||2\hat{\nu}(a_{i,k,n}-a_{j,k,n})|}{|i-j|}\\&\le 4\hat{w}\hat{\nu}\max_{1\le k\le M}\max_{1\le i,j\le M,i\neq j}\frac{|a_{i,k,n}-a_{j,k,n}|}{|i-j|}\le\tilde{w}\max_{1\le i,j\le M,i\neq j}\frac{|x_i-x_j|}{|i-j|}=\tilde{w}h_x.\end{aligned} \tag{51}$$

Hence, by Lemma 3.3, $\mathbf{B}_x-2\mu(c_0,\tilde{w},1)h_x^{\alpha}\mathbf{I}_{M^2}\prec\tilde{\mathbf{H}}_3\prec 4\hat{w}^2\mathbf{B}_x+\mu(c_0,\tilde{w},\hat{w}^2)h_x^{\alpha}\mathbf{I}_{M^2}$. Equation (46) and $\hat{c}\ge\check{c}>0$ imply that

$$\tau\le\frac{1}{8\hat{w}\hat{c}\hat{\nu}\mu(c_0,\tilde{w},1)}\le\frac{\hat{w}^2}{8\hat{c}\hat{\nu}\mu(c_0,\tilde{w},\hat{w}^2)}.$$

Moreover, by assumption (i), we have $\nu_{1,n}\in[0,\hat{\nu}]$ and

$$\begin{aligned}\check{a}=\frac{\check{c}}{1+2\hat{\nu}\check{c}}\le\frac{\check{c}}{1+(\nu_{1,n}+\nu_{2,n})\check{c}}&\le\frac{1}{M^2}\sum_{i,j=1}^{M}\frac{a_{i,j,n}}{1+(\nu_{1,n}+\nu_{2,n})a_{i,j,n}}\\&=\bar{a}\le\frac{1}{M^2}\sum_{i,j=1}^{M}a_{i,j,n}\le\hat{c}.\end{aligned} \tag{52}$$

Hence,

$$\eta_x\nu_{1,n}\check{a}\mathbf{B}_x-(4\hat{w})^{-1}\mathbf{I}_{M^2}\prec\eta_x\nu_{1,n}\bar{a}\tilde{\mathbf{H}}_3\prec 4\eta_x\nu_{1,n}\hat{c}\hat{w}^2\mathbf{B}_x+8^{-1}\hat{w}^2\mathbf{I}_{M^2}. \tag{53}$$

Similar to (51), by assumption (ii), one can prove that $\nabla_2(\mathbf{W}_n^2)\leq\tilde{w}h_y$. Applying Lemma 3.4 to (50), $\nabla_2(\mathbf{W}_n^2)\leq\tilde{w}h_y$, and assumption (iii) yields

$$\mathbf{B}_y-2\omega(c_0,\tilde{w},1)h_y^{\beta}\mathbf{I}_{M^2}\prec\tilde{\mathbf{H}}_4\prec 4\hat{w}^2\mathbf{B}_y+\omega(c_0,\tilde{w},\hat{w}^2)h_y^{\beta}\mathbf{I}_{M^2}.$$

Equation (46) and $\hat{c}\geq\check{c}>0$ imply that

$$\tau\leq\frac{1}{8\hat{\nu}\hat{w}\hat{c}\omega(c_0,\tilde{w},1)}\leq\frac{\hat{w}^2}{8\hat{\nu}\hat{c}\omega(c_0,\tilde{w},\hat{w}^2)}.$$

Hence, $\eta_y\mathbf{B}_y-(4\hat{\nu}\hat{w}\hat{c})^{-1}\mathbf{I}_{M^2}\prec\eta_y\tilde{\mathbf{H}}_4\prec 4\eta_y\hat{w}^2\mathbf{B}_y+(8\hat{\nu}\hat{c})^{-1}\hat{w}^2\mathbf{I}_{M^2}$, which together with (52) and $\nu_{2,n}\in[0,\hat{\nu}]$ implies that

$$\eta_y\nu_{2,n}\check{a}\mathbf{B}_y-(4\hat{w})^{-1}\mathbf{I}_{M^2}\prec\eta_y\nu_{2,n}\bar{a}\tilde{\mathbf{H}}_4\prec 4\eta_y\nu_{2,n}\hat{c}\hat{w}^2\mathbf{B}_y+8^{-1}\hat{w}^2\mathbf{I}_{M^2}. \tag{54}$$

By (54) and (53),

$$\bar{\theta}_n(\bar{\theta}_n\mathbf{W}_n^2-2^{-1}\hat{w}^{-1}\mathbf{I}_{M^2}+\check{a}\mathbf{B})\prec\tilde{\mathbf{Q}}_{p1}\prec\bar{\theta}_n(\bar{\theta}_n\mathbf{W}_n^2+4^{-1}\hat{w}^2\mathbf{I}_{M^2}+4\hat{c}\hat{w}^2\mathbf{B}).$$

Moreover, by assumption (i), we have $\hat{w}^{-1}\leq\bar{\theta}_n\leq 1$ and $\mathbf{I}_{M^2}\preceq\mathbf{W}_n^2\preceq\hat{w}^2\mathbf{I}_{M^2}$. Hence, we get $(2\hat{w}^2)^{-1}\mathbf{I}_{M^2}+\hat{w}^{-1}\check{a}\mathbf{B}\prec\tilde{\mathbf{Q}}_{p1}\prec(5/4)\hat{w}^2\mathbf{I}_{M^2}+4\hat{c}\hat{w}^2\mathbf{B}$, which together with (45) implies that

$$\mathbf{O}\prec(2\hat{w}^2)^{-1}\mathbf{I}_{M^2}+\hat{w}^{-1}\check{a}\mathbf{B}+\tilde{\mathbf{Q}}_{p2}\prec\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\prec(5/4)\hat{w}^2\mathbf{I}_{M^2}+4\hat{c}\hat{w}^2\mathbf{B}+\tilde{\mathbf{Q}}_{p2}. \tag{55}$$

For any nonzero vector $\mathbf{y}\in\mathbb{R}^{M^2\times 1}$, denote $\mathbf{z}=\mathbf{P}_n^{-1}\mathbf{y}$. Then, it holds that

$$\frac{\mathbf{y}^{\mathrm{T}}(\mathbf{A}_n\mathbf{P}_n^{-1})^{\mathrm{T}}(\mathbf{A}_n\mathbf{P}_n^{-1})\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}=\frac{\mathbf{z}^{\mathrm{T}}\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{z}}.$$

By (i), (52), and the fact that both $\tilde{\mathbf{Q}}_{a2}$ and $\tilde{\mathbf{Q}}_{p2}$ are real symmetric positive semidefinite, it is easy to check that

$$\frac{\check{a}^2}{\hat{c}^2}\leq\frac{\mathbf{z}^{\mathrm{T}}\tilde{\mathbf{Q}}_{a2}\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\tilde{\mathbf{Q}}_{p2}\mathbf{z}}\leq\frac{\hat{c}^2}{\check{a}^2}\qquad\text{or}\qquad\mathbf{z}^{\mathrm{T}}\tilde{\mathbf{Q}}_{a2}\mathbf{z}=\mathbf{z}^{\mathrm{T}}\tilde{\mathbf{Q}}_{p2}\mathbf{z}=0. \tag{56}$$

Applying Proposition 2.4 to (49), (55), and (56) yields

$$\frac{\mathbf{z}^{\mathrm{T}}\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{z}}\leq\frac{\mathbf{z}^{\mathrm{T}}[(5/4)\mathbf{I}_{M^2}+4\hat{c}\mathbf{B}+\tilde{\mathbf{Q}}_{a2}]\mathbf{z}}{\mathbf{z}^{\mathrm{T}}[(2\hat{w}^2)^{-1}\mathbf{I}_{M^2}+\hat{w}^{-1}\check{a}\mathbf{B}+\tilde{\mathbf{Q}}_{p2}]\mathbf{z}}\leq\max\left\{\frac{5\hat{w}^2}{2},\frac{4\hat{c}\hat{w}}{\check{a}},\frac{\hat{c}^2}{\check{a}^2}\right\}=\hat{s} \tag{57}$$

and

$$\frac{\mathbf{z}^{\mathrm{T}}\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{P}_n^{\mathrm{T}}\mathbf{P}_n\mathbf{z}}\geq\frac{\mathbf{z}^{\mathrm{T}}[2^{-1}\mathbf{I}_{M^2}+\check{c}\mathbf{B}+\tilde{\mathbf{Q}}_{a2}]\mathbf{z}}{\mathbf{z}^{\mathrm{T}}[(5/4)\hat{w}^2\mathbf{I}_{M^2}+4\hat{c}\hat{w}^2\mathbf{B}+\tilde{\mathbf{Q}}_{p2}]\mathbf{z}}\geq\min\left\{\frac{2}{5\hat{w}^2},\frac{\check{c}}{4\hat{c}\hat{w}^2},\frac{\check{a}^2}{\hat{c}^2}\right\}=\check{s}. \tag{58}$$

The result follows from (57)–(58). ∎

Remark 3.6. In section 5, we numerically demonstrate that the condition number of $\mathbf{A}_n$ arising from two-dimensional fractional diffusion equations depends on η_x and η_y, i.e., $\mathbf{A}_n$ can be ill-conditioned when η_x or η_y is large. On the other hand, Theorems 3.1, 3.2, and 3.5 show that the condition number of the precondition matrix $\mathbf{A}_n\mathbf{P}_n^{-1}$ is bounded by a positive constant $\sqrt{\hat{s}/\check{s}}$ when the coefficient functions satisfy the related assumptions. Hence, the splitting preconditioning technique improves the condition number of $\mathbf{A}_n$ in the two-dimensional case. Especially, when the conjugate gradient method is employed to solve the normalized preconditioned system, the method converges linearly within an iteration number independent of τ, h_x, and h_y.

4. Numerical schemes for Riesz derivative. In the previous two sections, we see that our theoretical analysis depends on two assumptions on the discretization scheme of the Riesz derivative (5). That is,

$$\mathbf{O} \prec \mathbf{S}_\alpha \qquad \forall \alpha \in (1,2), \tag{59}$$

$$\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha \qquad \forall \alpha \in (1,2). \tag{60}$$

Here we verify several numerical schemes proposed in [20, 27, 30, 8, 6] and show that those schemes satisfy the assumptions (59)–(60).

Let us first introduce some notations. Let $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ denote two sequences. For some nonnegative integer m, define mappings $\mathcal{S}_{\pm m}(\cdot)$, $\mathcal{F}_{\pm m}(\cdot)$, respectively, as

$$\begin{aligned}
\{b_k\}_{k\geq 0} = \mathcal{S}_m(\{a_k\}_{k\geq 0}) &\Longleftrightarrow b_k = a_{k+m}, \quad k \geq 0,\\
\{b_k\}_{k\geq 0} = \mathcal{S}_{-m}(\{a_k\}_{k\geq 0}) &\Longleftrightarrow b_k = 0,\ 0 \leq k \leq m-1 \ \text{ and } \ b_k = a_{k-m}, \quad k \geq m,\\
\{b_k\}_{k\geq 0} = \mathcal{F}_m(\{a_k\}_{k\geq 0}) &\Longleftrightarrow b_k = a_{m-k},\ 0 \leq k \leq m \ \text{ and } \ b_k = 0, \quad k > m.\\
\{b_k\}_{k\geq 0} = \mathcal{F}_{-m}(\{a_k\}_{k\geq 0}) &\Longleftrightarrow b_k = a_k,\ 0 \leq k \leq -m \ \text{ and } \ b_k = 0, \quad k > -m.
\end{aligned}$$

For any sequences $\{a_k\}_{k\geq 0}$ and for some integer m, define the operator

$$\mathcal{R}_m(\{a_k\}_{k\geq 0}) = \sigma_\alpha[\mathcal{S}_m(\{a_k\}_{k\geq 0}) + \mathcal{F}_m(\{a_k\}_{k\geq 0})].$$

4.1. Verification of schemes from [20, 30]. In this subsection, we verify the conditions (59) and (60) for the first-order shifted Grünwald formula from [20] and two second-order weighted-shifted Grünwald formulas from [30]. Let

$$g_0^{(\alpha)} = -1, \quad g_{k+1}^{(\alpha)} = \left(1 - \frac{\alpha+1}{k+1}\right) g_k^{(\alpha)}, \quad k = 0, 1, 2, \ldots. \tag{61}$$

Lemma 4.1 (see [30, 31]).

(i) $g_1^{(\alpha)} > 0,\ g_0^{(\alpha)} < g_2^{(\alpha)} < g_3^{(\alpha)} < \cdots \leq 0, \quad \sum_{k=0}^{\infty} g_k^{(\alpha)} = 0;$

(ii) $\{g_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$.

Theorem 4.2. *A first-order scheme for* (5) *resulting from* [20] *can be expressed as*

$$\{s_k^{(\alpha)}\}_{k\geq 0} = \mathcal{R}_1(\{g_k^{(\alpha)}\}_{k\geq 0}),$$

which satisfies $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *and* $\mathbf{O} \prec \mathbf{S}_\alpha$ *for any* $\alpha \in (1,2)$.

Proof. Since $\{g_k^{(\alpha)}\}_{k\geq 0}\in\mathcal{D}_\alpha$, it is easy to see that $\mathcal{S}_1(\{g_k^{(\alpha)}\}_{k\geq 0})\in\mathcal{D}_\alpha$. Moreover, since $\mathcal{F}_1(\{g_k^{(\alpha)}\}_{k\geq 0})$ has finite nonzero elements, $\mathcal{F}_1(\{g_k^{(\alpha)}\}_{k\geq 0})\in\mathcal{D}_\alpha$. As mentioned above, $\mathcal{D}_\alpha$ is a linear space. Hence, $\{s_k^{(\alpha)}\}_{k\geq 0}=\mathcal{R}_1(\{g_k^{(\alpha)}\}_{k\geq 0})\in\mathcal{D}_\alpha$.

By (i), it is easy to check that the so defined $\mathbf{S}_\alpha$ is strictly diagonally dominant with positive diagonal entries. Moreover, $\mathbf{S}_\alpha$ is Hermitian. Hence, by the Gershgorin circle theorem, $\mathbf{O}\prec\mathbf{S}_\alpha$, which completes the proof. ∎

THEOREM 4.3. *Two second-order schemes for* (5) *resulting from* [30] *can be expressed as follows:*

(i) $\{s_k^{(\alpha)}\}_{k\geq 0}=\mathcal{R}_1\big(\frac{\alpha}{2}\{g_k^{(\alpha)}\}_{k\geq 0}+\frac{2-\alpha}{2}\mathcal{S}_{-1}(\{g_k^{(\alpha)}\}_{k\geq 0})\big)$,

(ii) $\{s_k^{(\alpha)}\}_{k\geq 0}=\mathcal{R}_1\big(\frac{2+\alpha}{4}\{g_k^{(\alpha)}\}_{k\geq 0}+\frac{2-\alpha}{4}\mathcal{S}_{-2}(\{g_k^{(\alpha)}\}_{k\geq 0})\big)$,

both of which satisfy $\{s_k^{(\alpha)}\}_{k\geq 0}\in\mathcal{D}_\alpha$ *and* $\mathbf{O}\prec\mathbf{S}_\alpha$ *for any* $\alpha\in(1,2)$.

Proof. The proof of $\{s_k^{(\alpha)}\}_{k\geq 0}\in\mathcal{D}_\alpha$ is similar to that of Theorem 4.2. The proof of $\mathbf{O}\prec\mathbf{S}_\alpha$ for $\mathbf{S}_\alpha$ resulting from both (i) and (ii) can be found directly in [30, Theorem 2]. ∎

4.2. Verification of schemes from [8]. In this subsection, we verify the conditions (59) and (60) for a series of second-, third-, and fourth-order schemes proposed in [8]. Let

$$q_k^{(\alpha)}=-\left(\frac{3}{2}\right)^{\alpha}\sum_{j=0}^{k}3^{-j}g_j^{(\alpha)}g_{k-j}^{(\alpha)},\quad k\geq 0,$$

with $g_j^{(\alpha)}(j\geq 0)$ given by (61).

LEMMA 4.4. $\{q_k^{(\alpha)}\}_{k\geq 0}\in\mathcal{D}_\alpha$.

Proof. By (ii) in Lemma 4.1, $\{g_k^{(\alpha)}\}_{k\geq 0}\in\mathcal{D}_\alpha$. Denote $c_0=||\{g_k^{(\alpha)}\}_{k\geq 0}||_{\mathcal{D}_\alpha}$ and $c_1=2^{\alpha+1}c_0$. Then, it holds that

$$|g_k^{(\alpha)}|\leq\frac{c_0}{(1+k)^{\alpha+1}}=c_0\left(1+\frac{1}{1+k}\right)^{\alpha+1}\frac{1}{(2+k)^{\alpha+1}}\leq\frac{c_1}{(2+k)^{\alpha+1}},\quad k\geq 0. \tag{62}$$

Denote $c_2 = c_1^2(3/2)^\alpha$. By (62),

$$
\begin{aligned}
|q_k^{(\alpha)}| &\le \left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{k} |g_j^{(\alpha)}||g_{k-j}^{(\alpha)}| \\
&\le c_2 \sum_{j=0}^{k} \frac{1}{(2+j)^{\alpha+1}(2+k-j)^{\alpha+1}} \\
&\le c_2 \sum_{j=0}^{k} \int_j^{j+1} \frac{1}{(1+x)^{\alpha+1}(2+k-x)^{\alpha+1}} dx \\
&= c_2 \left(\int_0^{\frac{k}{2}} + \int_{\frac{k}{2}}^{k+1} \right) \frac{1}{(1+x)^{\alpha+1}(2+k-x)^{\alpha+1}} dx \\
&\le \frac{c_2}{(1+\frac{k}{2})^{\alpha+1}} \left[\int_0^{\frac{k}{2}} \frac{1}{(1+x)^{\alpha+1}} dx + \int_{\frac{k}{2}}^{k+1} \frac{1}{(2+k-x)^{\alpha+1}} dx \right] \\
&= \frac{c_2[1 + 2^{-\alpha} - (1+\frac{k}{2})^{-\alpha} - (2+\frac{k}{2})^{-\alpha}]}{\alpha(1+\frac{k}{2})^{\alpha+1}} \\
&\le \frac{c_2(2^{\alpha+1}+2)}{\alpha(1+k)^{\alpha+1}}, \quad k \ge 0,
\end{aligned}
$$

which implies that $||\{q_k^{(\alpha)}\}_{k\ge0}||_{\mathcal{D}_\alpha} < +\infty$. The proof is complete. ∎

THEOREM 4.5. *A series of second-, third-, and fourth-order schemes for* (5) *resulting from* [8] *has the following form (see* [8]*)*

$$\{s_k^{(\alpha)}\}_{k\ge0} = \sum_{j=m_1}^{m_2} a_j \mathcal{R}_j(\{q_k^{(\alpha)}\}_{k\ge0}),$$

where $a_j (m_1 \le j \le m_2)$ are some specified real constants and m_2 and m_1 are some specified integer constants such that $m_2 > m_1$. The so defined $\{s_k^{(\alpha)}\}_{k\ge0}$ satisfies $\{s_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$ and $\mathbf{O} \prec \mathbf{S}_\alpha$ for any $\alpha \in (1,2)$.

Proof. Because $a_j (m_1 \le j \le m_2)$, m_1, m_2 are all constants, and $\{q_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$, similar to the proof of Theorem 4.2, it is easy to check that $\{s_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$. The proof of $\mathbf{O} \prec \mathbf{S}_\alpha$ can be found directly in [8, Theorems 2.10–2.12]. ∎

4.3. Verification of scheme from [6]. In this subsection, we verify conditions (59) and (60) for the second-order fractional central difference scheme [6], whose coefficients are defined as follows:

$$s_k^{(\alpha)} = \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(\frac{\alpha}{2} - k + 1)\Gamma(\frac{\alpha}{2} + k + 1)}, \qquad k \ge 0. \tag{63}$$

THEOREM 4.6 (see [6]). *The second-order scheme* (63) *satisfies $\{s_k^{(\alpha)}\}_{k\ge0} \in \mathcal{D}_\alpha$ and $\mathbf{O} \prec \mathbf{S}_\alpha$ for any $\alpha \in (1,2)$.*

4.4. Verification of scheme from [27]. In this subsection, we verify conditions (59) and (60) for the second-order scheme from [27]. Let

$$
\begin{aligned}
&p_0^{(\alpha)} = -1, \quad p_1^{(\alpha)} = 4 - 2^{3-\alpha}, \quad p_2^{(\alpha)} = -3^{3-\alpha} + 4 \times 2^{3-\alpha} - 6, \\
&p_k^{(\alpha)} = -(k+1)^{3-\alpha} + 4k^{3-\alpha} - 6(k-1)^{3-\alpha} + 4(k-2)^{3-\alpha} - (k-3)^{3-\alpha}, \quad k \ge 3.
\end{aligned}
$$

LEMMA 4.7 (see [9, Lemma 3.2])). $\sum_{j=0}^{+\infty} p_k^{(\alpha)} = 0$, $p_1^{(\alpha)} > 0$, $p_0^{(\alpha)} + p_2^{(\alpha)} < 0$, $p_k^{(\alpha)} \leq 0$ *for* $k \geq 3$.

LEMMA 4.8. $\{p_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$.

Proof. Define

$$v(x) = x^{3-\alpha}, \quad \Delta_k = v(k+1) - 2v(k) + v(k-1), \quad k \geq 1.$$

By Lemma 4.7, $p_k^{(\alpha)} \leq 0$ for $k \geq 3$. Thus,

$$|p_k^{(\alpha)}| = -p_k^{(\alpha)} = \Delta_k - 2\Delta_{k-1} + \Delta_{k-2}, \quad k \geq 3.$$

By Taylor expansion, it holds that

$$\Delta_k = v^{(2)}(k) + w_k, \quad w_k = \int_k^{k+1} \frac{v^{(4)}(\xi)(k+1-\xi)^3}{3!} d\xi + \int_{k-1}^{k} \frac{v^{(4)}(\xi)(1+\xi-k)^3}{3!} d\xi.$$

Moreover,

$$|w_k| \leq \frac{1}{3!}\left(\int_k^{k+1} |v^{(4)}(\xi)| d\xi + \int_{k-1}^{k} |v^{(4)}(\xi)|\right) \leq c_1(k-1)^{-1-\alpha}, \quad k \geq 2,$$

with $c_1 = \frac{\Gamma(4-\alpha)}{3!\Gamma(-\alpha)}$. Using Taylor expansion again,

$$v^{(2)}(k) - 2v^{(2)}(k-1) + v^{(2)}(k-2) = v^{(4)}(k-1) + a_k, \quad k \geq 2,$$

with

$$a_k = \int_{k-1}^{k} \frac{v^{(6)}(\xi)(k-\xi)^3}{3!} d\xi + \int_{k-2}^{k-1} \frac{v^{(6)}(\xi)(2+\xi-k)^3}{3!} d\xi, \quad k \geq 2.$$

Moreover,

$$|a_k| \leq \frac{1}{3!}\left(\int_{k-1}^{k} |v^{(6)}(\xi)| d\xi + \int_{k-2}^{k-1} |v^{(6)}(\xi)| d\xi\right) \leq c_2(k-2)^{-3-\alpha}, \quad k \geq 3,$$

with $c_2 = \frac{\Gamma(4-\alpha)}{3!\Gamma(-\alpha-2)}$. Therefore,

$$\begin{aligned}
|p_k^{(\alpha)}| &= \Delta_k - 2\Delta_{k-1} + \Delta_{k-2} \\
&= v^{(2)}(k) - 2v^{(2)}(k-1) + v^{(2)}(k-2) + w_k - 2w_{k-1} + w_{k-2} \\
&= v^{(4)}(k-1) + a_k + w_k - 2w_{k-1} + w_{k-2} \\
&\leq c_2(k-2)^{-3-\alpha} + c_1\left[2(k-1)^{-1-\alpha} + 2(k-2)^{-1-\alpha} + (k-3)^{-1-\alpha}\right] \\
&\leq \frac{c_2 + 5c_1}{(k-3)^{1+\alpha}} = \frac{(c_2 + 5c_1)(1+k)^{1+\alpha}}{(k-3)^{1+\alpha}(1+k)^{1+\alpha}} \leq \frac{5(c_2 + 5c_1)}{(1+k)^{1+\alpha}}, \quad k \geq 4,
\end{aligned}$$

which implies that $\{p_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$. ∎

THEOREM 4.9. *Denote* $c_\alpha = [\Gamma(4-\alpha)]^{-1}$. *A second-order scheme for* (5) *resulting from* [27] *can be expressed as*

$$\{s_k^{(\alpha)}\}_{k\geq 0} = \mathcal{R}_1(c_\alpha \{p_k^{(\alpha)}\}_{k\geq 0}),$$

which satisfies $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ *and* $\mathbf{O} \prec \mathbf{S}_\alpha$ *for any* $\alpha \in (1, 2)$.

Proof. By Lemma 4.7, it is easy to see that the so defined $\mathbf{S}_\alpha$ is strictly diagonally dominant with positive diagonal entries, which therefore holds that $\mathbf{O} \prec \mathbf{S}_\alpha$. Moreover, $\{s_k^{(\alpha)}\}_{k\geq 0} \in \mathcal{D}_\alpha$ follows from Lemma 4.8. ∎

Remark 4.10. Actually, most of the existing unconditionally stable discretization schemes of form (5) satisfy the conditions (59) and (60). We only verify several representative ones among these schemes in this section to demonstrate that the conditions (59)–(60) are general enough.

5. Numerical results. In this section, we use several examples to test the performance of the proposed splitting preconditioner and compare it with other preconditioners and other solvers. All numerical experiments are performed via MATLAB R2013a on a PC with the configuration Intel(R) Core(TM) i5-4590 CPU 3.30 GHz and 8 GB RAM.

For convenience, we use **SP** to denote the splitting preconditioner. Implementation of the preconditioner consists of inversion of $\mathbf{W}_n$ and $\mathbf{T}_n$. $\mathbf{W}_n^{-1}$ is just a diagonal matrix. Moreover, as $\mathbf{T}_n$ is a real symmetric positive definite Toeplitz matrix, $\mathbf{T}_n^{-1}$ can be expressed explicitly in the Gohberg–Semencul-type formula [11]. In detail, $\mathbf{T}_n^{-1}$ can be expressed explicitly in terms of skew-circulant and circulant matrices generated by a vector $\mathbf{v}$, where $\mathbf{v}$ is solution of the linear system $\mathbf{T}_n\mathbf{v} = \mathbf{e}_1$, with $\mathbf{e}_1$ being the first column of the identity matrix; see [11]. We can employ the fast direct Toeplitz solver [22] or the fast multigrid method [7] to solve $\mathbf{T}_n\mathbf{v} = \mathbf{e}_1$. With the Gohberg–Semencul-type formula, matrix-vector multiplication of the preconditioned matrix requires $\mathcal{O}(M \log M)$ operation and $\mathcal{O}(M)$ storage.

In the two-dimensional case, the implementation of the splitting preconditioner requires solving some two-level Toeplitz linear systems with the two-level-Toeplitz coefficient matrix $\mathbf{T}_n$. As suggested in [29, 3, 2], the algebraic multigrid (AMG) method is an efficient solver for such linear systems, for which we employ the AMG solver to implement the two-dimensional splitting preconditioner. For the choice of coarse-grid matrices, interpolation, and restriction in the AMG solver, we refer to the Garlerkin coarsening, piecewise linear interpolation and its transpose. Besides these components, the AMG solver also requires suitable choices of pre- and postsmoothing iterations. For a two-level Toeplitz linear system $\mathbf{Tx} = \mathbf{y}$ with $\mathbf{T}$ being a two-level-Toeplitz matrix, its smoothing iteration has a general form $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{R}^{-1}(\mathbf{y} - \mathbf{Tx}^k)$, where $\mathbf{R}$ is an approximation to $\mathbf{T}$, and $\mathbf{x}^k$ is an initial guess of $\mathbf{x}$. Thus, a good smoothing iteration is typically equipped with a special $\mathbf{R}$ which is easily invertible while approximating $\mathbf{T}$ as well as possible. With these considerations in mind, in the presmoothing stage, a suitable choice one can catch intuitively is to take $\mathbf{R} = \mathbf{T}_x$, with $\mathbf{T}_x$ being the block diagonal part of $\mathbf{T}$, which leads to the block Jacobi presmoothing iteration. Indeed, $\mathbf{T}_x$ is fast invertible with the help of the Gohberg–Semencul-type formula as discussed above. Moreover, compared with the diagonal part of $\mathbf{T}$ which leads to the Jacobi iteration, $\mathbf{T}_x$ is at least a better approximation to $\mathbf{T}$ from the perspectives of matrix structure, spectral variety, and norm of the error matrix. Nevertheless, we note that although the information of the fractional derivative along the x-direction is localized in the block diagonal part of $\mathbf{T}$ (e.g., $\eta_x \bar{d}_n \mathbf{B}_x$ in $\mathbf{T}_n$), the information of the fractional derivative along the y-direction is evenly distributed in $\mathbf{T}$ (e.g., $\eta_y \bar{e}_n \mathbf{B}_y$ in $\mathbf{T}_n$). That means $\mathbf{T}_x$ characterizes the derivative along the x-direction well, yet it is insufficient to characterize the derivative along the y-direction. To remedy the situation, we take $\mathbf{R} = \hat{\mathbf{P}}^{\mathrm{T}}\mathbf{T}_y\hat{\mathbf{P}}$ in the postsmoothing stage, where $\hat{\mathbf{P}}$ is a permutation matrix defined in (41) and $\mathbf{T}_y$ is the block diagonal part of $\hat{\mathbf{P}}\mathbf{T}\hat{\mathbf{P}}^{\mathrm{T}}$. The role of the permutation matrix $\hat{\mathbf{P}}$ is to rearrange the linear system from x-dominant ordering to y-dominant ordering. That means the information of the derivative along the y-direction in $\mathbf{T}$ is contained in $\hat{\mathbf{P}}^{\mathrm{T}}\mathbf{T}_y\hat{\mathbf{P}}$, which is regarded as a compensation of $\mathbf{T}_x$. Actually, the postsmoothing iteration chosen here is simply another block Jacobi iteration for the linear system rearranged into y-dominant ordering. We hope that such defined pre- and postsmoothing iterations could complement each other well and thus reduce the error efficiently. To save the operation cost, both the pre- and postsmoothing iterations are performed one time, respectively, in each V-cycle iteration. Moreover, the V-cycle iteration is performed only one time in each matrix-vector multiplication of the preconditioned matrix. By using the so defined AMG solver, matrix-vector multiplication of the preconditioned matrix only requires $\mathcal{O}(M^2 \log M)$ operation and $\mathcal{O}(M^2)$ storage.

Other testing preconditioners for (7) and (23) are listed as follows. The circulant preconditioner [17] and the multilevel circulant preconditioner [15] can be used to precondition (7) and (23), respectively. For convenience, we use $\mathbf{C}$ to denote the (multilevel) circulant preconditioner. FFTs are used to compute the corresponding preconditioned matrix-vector multiplication. Denote by $\mathbf{P}(k)$ the approximate inverse preconditioner [23] with k interpolating points for (7) while with k interpolating points in both the x- and y-directions, respectively, for (23); see [23]. FFTs are used to compute the corresponding preconditioned matrix-vector multiplication. Denote by $\mathbf{B}(k)$ the banded preconditioner of bandwidth k for $\mathbf{A}_n$ from (7) or (23); see [14]. Banded solvers are used to compute the corresponding preconditioned matrix-vector multiplication. Also, denote by $\mathbf{S}_1$ and $\mathbf{S}_2$ the two structure preserving preconditioners proposed in [10], for which the one-dimensional implementation is already discussed in [10]. A two-dimensional extension of $\mathbf{S}_1$ and $\mathbf{S}_2$ can be defined as

$$\mathbf{S}_1 = \mathbf{I}_{M^2} + (\eta_x\sigma_\alpha\mathbf{D}_n + \eta_y\sigma_\beta\mathbf{E}_n)\mathbf{H}, \quad \mathbf{S}_2 = \mathbf{I}_{M^2} + 2(\eta_x\sigma_\alpha\mathbf{D}_n + \eta_y\sigma_\beta\mathbf{E}_n)\mathbf{H},$$
$$\mathbf{H} = \mathbf{I}_M \otimes \mathbf{L}_M + \mathbf{L}_M \otimes \mathbf{I}_M, \quad \mathbf{L}_M = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{M\times M}.$$

The $\mathbf{S}_1$ and $\mathbf{S}_2$ defined above can be implemented using the same multigrid method as the one used for implementation of the two-dimensional splitting preconditioner.

We employ the preconditioned generalized minimal residual (PGMRES) method with different preconditioners to solve linear systems (7) and linear systems (23). Also, denote GMRES-**SP**, GMRES-**C**, GMRES-$\mathbf{P}(k)$, GMRES-$\mathbf{B}(k)$, GMRES-$\mathbf{S}_1$, GMRES-$\mathbf{S}_2$, PGMRES with preconditioners, **SP**, **C**, $\mathbf{P}(k)$, $\mathbf{B}(k)$, and $\mathbf{S}_1$, $\mathbf{S}_2$, respectively. Especially, we denote by GMRES-**I** the GMRES iteration without preconditioner. Moreover, a multigrid method with tridiagonal splitting iterations as smoothers proposed in [18] can also be used to solve the linear systems (7) and (23). We denote the V-cycle multigrid method with the tridiagonal splitting smoother proposed in [18] by MGM-**TS**. In the implementation of MGM-**TS**, both of the pre- and postsmoothing iterations are performed only one time in each V-cycle iteration. For all of these methods, we set the zero vector as the initial guess and set $\frac{||\mathbf{r}_k||_2}{||\mathbf{r}_0||_2} \leq$1e-7 as the stopping criterion, where $\mathbf{r}_k$ denotes residual vector in the kth iteration. All PGMRES methods tested here are restarted versions with a restarting number, 300.

For the choice of discretization scheme (5), we refer to scheme (i) of Theorem 4.3, which satisfies those theoretical requirements imposed in sections 2 and 3. Since there are N linear systems in (7) or (23) to be solved, N iteration numbers will be generated by above iterative solvers. We use "iter" to denote the average of these iteration numbers. Denote by "CPU" the running time, units of which are "s" for second and "h" for hour, respectively. Denote by aEb the number $a \times 10^b$. Define the relative error

$$\mathrm{E}_{M,N} = \frac{||\mathbf{u} - \tilde{\mathbf{u}}||_\infty}{||\mathbf{u}||_\infty},$$

where $\mathbf{u}$ and $\tilde{\mathbf{u}}$ denote the exact solution of the SFDE and the iterative solution of linear systems (7) or (23) deriving from iterative solvers, respectively.

Example 5.1. Consider the one-dimensional space-fractional diffusion equation (1)–(3) with

$$u(x,t) = t^2x^4(2-x)^4, \quad d(x,t) = (1+t)\exp(0.8x+12), \quad T = 1, \quad [x_l, x_R] = [0,2]$$
$$f(x,t) = 2tx^4(2-x)^4 - \sigma_\alpha d(x,t)t^2 \sum_{i=5}^{9} \frac{q_i\Gamma(i)\left[x^{i-1-\alpha} + (2-x)^{i-1-\alpha}\right]}{\Gamma(i-\alpha)},$$
$$q_5 = 16, \quad q_6 = -32, \quad q_7 = 24, \quad q_8 = -8, \quad q_9 = 1.$$

The $d(x,t)$ in Example 5.1 satisfies the theoretical assumptions of Theorem 2.5. We solve Example 5.1 by different solvers and list the results in Tables 1–4. Since $\mathrm{E}_{M,N}$ of the different solvers are all small and the same, results of $\mathrm{E}_{M,N}$ are not listed in the tables. For $\mathbf{P}(k)$ and $\mathbf{B}(k)$, only the results of $\mathbf{P}(5)$ and $\mathbf{B}(15)$ are shown, since the results by using the other values of k of $\mathbf{B}(k)$ or $\mathbf{P}(k)$ do not make a difference. Tables 1–4 show the performance of different solvers for different values of M and N, from which we see that the performance of the proposed solver, GMRES-**SP**, is better than that of other solvers in terms of both iterations and computational times.

To further illustrate the effectiveness of the splitting preconditioner, we also list the condition numbers of the coefficient matrix and the preconditioned matrix by the splitting preconditioner at the final time level for $N = 1$ and different values of $\eta = \tau/h^\alpha$ in Table 5. In this example, $\tau = 1/N$ and $h = 2/(M+1)$ are the temporal and spatial discretization sizes, respectively, and thus $\eta = (M+1)^\alpha/(2^\alpha N)$. Comparing the condition number of $\mathbf{A}_N$ in Table 5, we see that the condition number of the coefficient matrix is almost linearly dependent on η, which is large when η is large. On the other hand, the condition number of the preconditioned matrix by the splitting preconditioner is always close to 1 and almost unchanged as η increases. That means the condition number of the preconditioned matrix is independent of τ and h, which is in accordance with the theoretical results. In addition, if η goes to zero, then the coefficient matrix $\mathbf{A}_N$ tends to the identity matrix and thus is well-conditioned. That means $\mathbf{A}_N$ with small η does not even need a preconditioner.

Table 1
Results of different solvers when $N = 2^7$ for Example 5.1.

		GMRES-**SP**		GMRES-**B**(15)		GMRES-$\mathbf{S}_1$		GMRES-$\mathbf{S}_2$	
α	$M+1$	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
1.1	2^{12}	3.0	1.29s	28.0	7.48s	60.0	18.46s	60.0	18.12s
	2^{13}	3.0	3.80s	35.0	23.49s	71.9	52.34s	71.9	52.02s
	2^{14}	3.0	7.58s	43.0	56.76s	84.8	121.59s	85.3	122.37s
1.5	2^{12}	3.0	1.28s	20.0	5.07s	34.0	7.85s	34.0	7.86s
	2^{13}	3.0	3.55s	24.0	15.85s	39.0	22.88s	39.0	22.80s
	2^{14}	3.0	7.42s	28.0	36.28s	44.0	50.83s	44.0	50.24s
1.9	2^{12}	3.0	1.26s	7.0	2.58s	10.0	1.93s	10.0	1.92s
	2^{13}	4.0	4.07s	8.0	7.09s	10.0	5.40s	10.0	5.46s
	2^{14}	4.0	8.53s	9.0	15.29s	11.0	11.35s	11.0	11.33s

TABLE 2
Results of different solvers when $N = 2^7$ for Example 5.1.

		MGM-**TS**		GMRES-**C**		GMRES-**P**(5)		GMRES-**I**	
α	$M+1$	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
1.1	2^{12}	5.0	3.22s	20.0	3.18s	12.0	2.42s	588.5	583.78s
	2^{13}	6.0	8.84s	20.0	7.24s	12.0	6.62s	1142.2	1877.80s
	2^{14}	6.0	18.46s	20.0	13.91s	12.1	13.26s	2135.3	6061.74s
1.5	2^{12}	5.0	3.19s	20.0	3.12s	12.2	2.47s	>4000	>1h
	2^{13}	5.0	7.60s	20.0	7.01s	12.3	6.72s	>8000	>3h
	2^{14}	5.0	15.79s	20.0	13.60s	13.1	14.16s	>16000	>9h
1.9	2^{12}	4.0	2.70s	19.0	2.92s	11.8	2.49s	>4000	>1h
	2^{13}	4.0	6.36s	19.0	6.71s	12.2	7.03s	>8000	>3h
	2^{14}	4.0	13.12s	19.0	12.81s	13.9	17.47s	>16000	>9h

TABLE 3
Results of different solvers when $M + 1 = 2^7$ for Example 5.1.

		GMRES-**SP**		GMRES-**B**(15)		GMRES-$\mathbf{S}_1$		GMRES-$\mathbf{S}_2$	
α	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
1.1	2^{12}	3.0	2.83s	8.0	6.14s	23.0	16.00s	23.0	15.93s
	2^{13}	3.0	5.72s	8.0	12.32s	23.0	31.88s	23.0	31.89s
	2^{14}	3.0	11.49s	8.0	24.63s	22.5	61.87s	23.0	63.63s
1.5	2^{12}	3.0	2.82s	7.0	5.49s	16.0	9.79s	16.0	9.78s
	2^{13}	3.0	5.70s	7.0	11.02s	16.0	19.57s	16.0	19.57s
	2^{14}	3.0	11.49s	7.0	21.95s	15.7	38.14s	16.0	39.19s
1.9	2^{12}	3.0	2.82s	4.0	3.77s	7.0	3.95s	7.0	3.94s
	2^{13}	3.0	5.71s	4.0	7.50s	7.0	7.90s	7.0	7.89s
	2^{14}	3.0	11.49s	4.0	14.97s	7.0	15.78s	7.0	15.75s

TABLE 4
Results of different solvers when $M + 1 = 2^7$ for Example 5.1.

		MGM-**TS**		GMRES-**C**		GMRES-**P**(5)		GMRES-**I**	
α	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
1.1	2^{12}	5.0	14.49s	18.0	11.82s	19.4	20.15s	58.0	68.44s
	2^{13}	5.0	29.02s	18.0	23.47s	19.4	40.11s	58.0	137.04s
	2^{14}	5.0	58.02s	18.0	47.23s	19.4	79.91s	58.0	274.29s
1.5	2^{12}	4.0	12.32s	18.0	11.84s	20.2	21.08s	87.0	137.92s
	2^{13}	4.0	24.64s	18.0	23.57s	20.2	42.09s	87.0	274.94s
	2^{14}	4.0	49.24s	18.0	47.09s	20.2	84.50s	87.0	551.59s
1.9	2^{12}	3.0	10.10s	18.0	11.74s	20.7	21.75s	122.0	253.64s
	2^{13}	3.0	20.22s	18.0	23.78s	20.9	43.99s	122.0	506.53s
	2^{14}	3.0	40.49s	18.0	47.30s	21.1	89.16s	122.0	1012.33s

TABLE 5
Condition numbers of $\mathbf{A}_N$ and the preconditioned matrix $\mathbf{A}_N\mathbf{P}_N^{-1}$ by the splitting preconditioner in Example 5.1 when η increases.

α	1.1			1.5			1.9		
$\eta = \frac{(M+1)^\alpha}{2^\alpha N}$	9.55E2	2.05E3	4.39E3	1.16E4	3.28E4	9.27E4	1.40E5	5.24E5	1.96E6
$\mathbf{A}_N$	2.71E3	5.83E3	1.25E4	3.56E4	1.01E5	2.87E5	5.12E5	1.92E6	7.21E6
$\mathbf{A}_N\mathbf{P}_N^{-1}$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01

Example 5.2. Consider the two-dimensional space-fractional diffusion equation (17)–(19) with

$$
\begin{aligned}
&u(x,t) = t^2 x^4 (2-x)^4 y^4 (2-y)^4, \quad [x_L, x_R] = [0,2], \quad [y_D, y_U] = [0,2], \quad T = 1,\\
&d(x,y,t) = (2+t)\exp((\sin(40x)+5)(\sin(40y)+5)),\\
&e(x,y,t) = \exp(\sin(t) + (\sin(40x)+5)(\sin(40y)+5)),\\
&f(x,y,t) = 2tx^4(2-x)^4 y^4 (2-y)^4\\
&\qquad - \sigma_\alpha t^2 y^4 (2-y)^4 d(x,y,t) \sum_{i=5}^{9} \frac{q_i \Gamma(i)\left[x^{i-1-\alpha} + (2-x)^{i-1-\alpha}\right]}{\Gamma(i-\alpha)}\\
&\qquad - \sigma_\beta t^2 x^4 (2-x)^4 e(x,y,t) \sum_{i=5}^{9} \frac{q_i \Gamma(i)\left[y^{i-1-\beta} + (2-y)^{i-1-\beta}\right]}{\Gamma(i-\beta)},\\
&q_5 = 16, \quad q_6 = -32, \quad q_7 = 24, \quad q_8 = -8, \quad q_9 = 1.
\end{aligned}
$$

The coefficient functions d and e provided in Example 5.2 satisfy the assumptions required by Theorem 3.5. We solve Example 5.2 by different solvers. The corresponding numerical results are listed in Tables 6–9. To be clear, $\mathrm{E}_{M,N}$ obtained by GMRES-**SP**, GMRES-**B**(15), GMRES-$\mathbf{S}_1$, and GMRES-$\mathbf{S}_2$ are almost the same and always less than 2×10^{-3} while $\mathrm{E}_{M,N}$ obtained from GMRES-**P**(5), GMRES-**C**, and GMRES-**I** is always larger than 1×10^{-1}. Especially, the notation, "*" for MGM-**TS** denotes its divergence. Bad performance of the four solvers GMRES-**P**(5), GMRES-**C**, GMRES-**I**, and MGM-**TS**, is due to the fact that the coefficient functions in Example 5.2 oscillate too much. According to Tables 6–9, the performance of the proposed solver, GMRES-**SP**, is better than that of other solvers in terms of both iterations and computational times. We remark that the convergence results by the splitting preconditioner are very good, although the coefficient functions in Example 5.2 oscillate much.

We list the condition numbers of the coefficient matrix and the preconditioned matrix by the splitting preconditioner at the final time level for $N = 1$ and different values of η_x and η_y in Table 10. In this example, $\tau = 1/N$, $h_x = 2/(M+1)$, $h_y = 2/(M+1)$, $\eta_x = \tau/h^\alpha = (M+1)^\alpha/(2^\alpha N)$, and $\eta_y = \tau/h^\beta = (M+1)^\beta/(2^\beta N)$. From Table 10, we see that the condition number of the coefficient matrix depends almost linearly on $\max\{\eta_x, \eta_y\}$, which is very ill-conditioned for even properly large values of $\max\{\eta_x, \eta_y\}$. Nevertheless, the condition number of the preconditioned matrix by the splitting preconditioner is always close to 1 for different values of η_x and η_y, which implies the condition number of the preconditioned matrix is independent of τ and h as suggested in Theorem 3.5. In addition, when $\max\{\eta_x, \eta_y\}$ tends to zero, the coefficient matrix $\mathbf{A}_N$ tends to identity, which is well-conditioned and thus does not need a preconditioner.

TABLE 6
Results of different solvers when $N = 2^4$ for Example 5.2.

(α,β)	$M+1$	GMRES-**SP**		GMRES-**B**(15)		GMRES-**P**(5)		GMRES-**C**	
		Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	5.0	1.21s	23.9	4.23s	>10000	>1 h	>12000	>1 h
	2^8	4.0	4.35s	34.3	23.82s	>20000	>8 h	>24000	>8 h
	2^9	4.0	23.80s	50.8	210.04s	>40000	>64 h	>36000	>64h
(1.1,1.6)	2^7	5.0	1.25s	29.1	4.73s	>10000	>1 h	>12000	>1 h
	2^8	4.0	4.27s	41.6	27.59s	>20000	>8 h	>24000	>8 h
	2^9	4.0	23.77s	61.9	276.72s	>40000	>64 h	>48000	>64 h
(1.1,1.9)	2^7	5.0	1.25s	29.6	5.09s	>10000	>1 h	>12000	>1 h
	2^8	4.4	4.55s	42.9	31.10s	>20000	>8 h	>24000	>8 h
	2^9	4.0	23.75s	62.8	291.92s	>40000	>64 h	>48000	>64 h
(1.3,1.3)	2^7	5.0	1.24s	28.8	4.88s	>10000	>1 h	>12000	>1 h
	2^8	5.0	5.00s	43.0	30.83s	>20000	>8 h	2413.9	2623.02s
	2^9	4.7	26.58s	65.9	314.59s	>40000	>64 h	>3500	>4h
(1.6,1.6)	2^7	5.0	1.25s	48.0	7.47s	>10000	>1 h	1521.9	494.49s
	2^8	5.0	5.03s	79.3	59.00s	>20000	>8 h	1507.0	1789.21s
	2^9	5.0	28.00s	126.7	844.22s	>40000	>64 h	874.7	9532.82s
(1.9,1.9)	2^7	6.0	1.49s	79.9	13.48s	>10000	>1 h	1280.8	391.15s
	2^8	6.0	5.74s	136.6	124.33s	>20000	>8 h	741.4	795.62s
	2^9	5.8	31.18s	255.9	2809.50s	>40000	>64 h	483.4	4771.28s

TABLE 7
Results of different solvers when $N = 2^4$ for Example 5.2.

(α,β)	$M+1$	MGM-**TS**		GMRES-$\mathbf{S_1}$		GMRES-$\mathbf{S_2}$		GMRES-**I**	
		Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	*	*	7775.2	3230.07s	8095.9	3377.95s	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h
(1.1,1.6)	2^7	*	*	>10000	>1h	>10000	>1h	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h
(1.1,1.9)	2^7	*	*	>10000	>1h	>10000	>1h	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h
(1.3,1.3)	2^7	*	*	7784.1	3271.15s	7824.6	3197.48s	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h
(1.6,1.6)	2^7	*	*	>10000	>1h	>10000	>1h	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h
(1.9,1.9)	2^7	*	*	>10000	>1h	>10000	>1h	>12000	>1 h
	2^8	*	*	>20000	>8h	>20000	>8h	>24000	>8 h
	2^9	*	*	>40000	>64h	>40000	>64h	>48000	>64h

TABLE 8
Results of different solvers when $M+1=2^6$ *for Example* 5.2.

		GMRES-**SP**		GMRES-**B**(15)		GMRES-**P**(5)		GMRES-**C**	
(α,β)	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	5.0	4.50s	16.4	6.37s	>4000	>1 h	1189.2	1248.57s
	2^8	5.0	8.91s	16.4	13.59s	>4000	>2 h	1189.3	2519.16s
	2^9	5.0	17.81s	16.4	27.09s	>4000	>4 h	1189.3	4979.84s
(1.1,1.6)	2^7	5.0	4.45s	20.0	7.12s	>4000	>1 h	1304.6	1322.10s
	2^8	5.0	8.91s	20.0	16.26s	>4000	>2 h	1304.3	2635.50s
	2^9	5.0	17.81s	20.0	30.53s	>4000	>4 h	1304.0	5290.56s
(1.1,1.9)	2^7	5.0	4.46s	20.6	8.24s	>4000	>1 h	1466.3	1515.81s
	2^8	5.0	8.92s	20.6	14.98s	>4000	>2 h	1467.6	3027.08s
	2^9	5.0	17.81s	20.6	32.06s	>4000	>4 h	1468.1	6064.78s
(1.3,1.3)	2^7	5.0	4.46s	19.5	7.97s	>4000	>1 h	1166.0	1204.20s
	2^8	5.0	8.91s	19.5	15.07s	>4000	>2 h	1166.1	2400.35s
	2^9	5.0	17.83s	19.5	30.31s	>4000	>4 h	1166.1	4807.09s
(1.6,1.6)	2^7	6.0	5.09s	29.0	10.68s	>4000	>1 h	1040.3	1026.84s
	2^8	6.0	10.18s	29.0	20.34s	>4000	>2 h	1040.5	2027.42s
	2^9	6.0	20.33s	29.0	41.55s	>4000	>4 h	1040.7	4038.67s
(1.9,1.9)	2^7	6.4	5.35s	43.5	15.68s	>4000	>1 h	544.3	538.44s
	2^8	6.4	10.68s	43.5	31.13s	>4000	>2 h	544.3	1060.72s
	2^9	6.4	21.37s	43.5	62.34s	>4000	>4 h	544.4	2065.63s

TABLE 9
Results of different solvers when $M+1=2^6$ *for Example* 5.2.

		MGM-**TS**		GMRES-$\mathbf{S_1}$		GMRES-$\mathbf{S_2}$		GMRES-**I**	
(α,β)	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	*	*	551.7	673.06s	551.9	641.93s	>8000	>2 h
	2^8	*	*	551.7	1329.61s	551.6	1279.39s	>8000	>4 h
	2^9	*	*	551.8	2625.82s	551.6	2572.02s	>8000	>8 h
(1.1,1.6)	2^7	*	*	951.0	1133.94s	950.8	1127.59s	>8000	>2 h
	2^8	*	*	950.4	2454.19s	950.9	2256.46s	>8000	>4 h
	2^9	*	*	949.2	4989.08s	950.2	4504.10s	>8000	>8 h
(1.1,1.9)	2^7	*	*	3119.8	3753.29s	3119.5	3735.37s	>8000	>2 h
	2^8	*	*	3120.0	7478.13s	3125.8	7554.73s	>8000	>4 h
	2^9	*	*	3109.0	14936.95s	3116.1	15005.77s	>8000	>8 h
(1.3,1.3)	2^7	*	*	519.7	580.38s	519.4	575.56s	>8000	>2 h
	2^8	*	*	520.4	1158.47s	520.2	1153.17s	>8000	>4 h
	2^9	*	*	520.1	2312.69s	520.4	2337.90s	>8000	>8 h
(1.6,1.6)	2^7	*	*	317.4	390.83s	317.3	392.54s	>8000	>2 h
	2^8	*	*	316.4	855.38s	315.6	784.32s	>8000	>4 h
	2^9	*	*	317.2	1715.38s	317.3	1572.06s	>8000	>8 h
(1.9,1.9)	2^7	*	*	568.7	751.24s	568.7	682.61s	>8000	>2 h
	2^8	*	*	568.9	1382.91s	569.5	1384.61s	>8000	>4 h
	2^9	*	*	569.0	2778.70s	568.8	2772.26s	>8000	>8 h

Table 10

Condition numbers of $\mathbf{A}_N$ and the preconditioned matrix $\mathbf{A}_N\mathbf{P}_N^{-1}$ by the splitting preconditioner in Example 5.2 for different values of η_x and η_y with $\eta_x = (M+1)^\alpha/(2^\alpha N)$ and $\eta_y = (M+1)^\beta/(2^\beta N)$.

$(\alpha,\beta)=(1.1,1.3)$				$(\alpha,\beta)=(1.1,1.6)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
9.85	14.93	2.01E9	1.01	9.85	27.86	3.17E9	1.01
21.11	36.76	6.75E9	1.02	21.11	84.45	1.27E10	1.01
45.25	90.51	1.84E10	1.02	45.25	256.00	4.30E10	1.01
$(\alpha,\beta)=(1.1,1.9)$				$(\alpha,\beta)=(1.3,1.3)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
9.85	51.98	5.92E9	1.01	14.93	14.93	2.44E9	1.01
21.11	194.01	2.94E10	1.01	36.76	36.76	8.75E9	1.02
45.25	724.08	1.28E11	1.01	90.51	90.51	2.52E10	1.02
$(\alpha,\beta)=(1.6,1.6)$				$(\alpha,\beta)=(1.9,1.9)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
27.86	27.86	4.63E9	1.01	51.98	51.98	9.16E9	1.01
84.45	84.45	2.04E10	1.01	194.01	194.01	4.93E10	1.01
256.00	256.00	7.32E10	1.01	724.08	724.08	2.22E11	1.01

For the examples tested above, the coefficient functions are strictly positive and satisfy the assumptions required in the theoretical analysis in section 2. To further demonstrate the applicability of the splitting preconditioner, in Example 5.3 we also employ the splitting preconditioner to solve a two-dimensional FSDE in which $d(x,y,t)$ has zeros at the boundary and d and e do not share a common part with the one stated in Theorem 3.5. Such coefficient functions do not satisfy any assumption required in the analysis in section 3.

Example 5.3. Consider the two-dimensional space-fractional diffusion equation (17)–(19) with

$$\begin{aligned}
&u(x,t) = t^2x^4(2-x)^4y^4(2-y)^4, \quad [x_L,x_R]=[0,2], \quad [y_D,y_U]=[0,2], \quad T=1,\\
&d(x,y,t) = 2000(1+t)[(x-1)^2+y(2-y)],\\
&e(x,y,t) = 2^{11+t}[2-\cos(\pi(x-1)/2)+\sin(\pi y/2)],\\
&f(x,y,t) = 2tx^4(2-x)^4y^4(2-y)^4\\
&\qquad -\sigma_\alpha t^2y^4(2-y)^4d(x,y,t)\sum_{i=5}^{9}\frac{q_i\Gamma(i)\left[x^{i-1-\alpha}+(2-x)^{i-1-\alpha}\right]}{\Gamma(i-\alpha)}\\
&\qquad -\sigma_\beta t^2x^4(2-x)^4e(x,y,t)\sum_{i=5}^{9}\frac{q_i\Gamma(i)\left[y^{i-1-\beta}+(2-y)^{i-1-\beta}\right]}{\Gamma(i-\beta)},\\
&q_5=16, \quad q_6=-32, \quad q_7=24, \quad q_8=-8, \quad q_9=1.
\end{aligned}$$

Clearly, d and e in Example 5.3 do not satisfy any assumption required by the theorems in section 3. We solve Example 5.3 by PGMRES with different preconditioners. The corresponding numerical results are listed in Tables 11–14. Since $\mathrm{E}_{M,N}$ of the different solvers are all small and the same, the results of $\mathrm{E}_{M,N}$ are not listed in the tables. From Tables 11–14, we see that the performance of the proposed solver, GMRES-**SP**, is generally better than that of other solvers in terms of both iterations and computational times. That means the splitting preconditioner may still be applicable even if the coefficient functions do not satisfy the theoretical assumptions, which demonstrates the robustness of the splitting preconditioner.

Also, we list the condition numbers of the coefficient matrix and the preconditioned matrix by the splitting preconditioner at the final time level for $N = 1$ and different values of η_x and η_y in Table 15. In this example, $\tau = 1/N$, $h_x = 2/(M+1)$, $h_y = 2/(M+1)$, $\eta_x = \tau/h^\alpha = (M+1)^\alpha/(2^\alpha N)$, and $\eta_y = \tau/h^\beta = (M+1)^\beta/(2^\beta N)$. From Table 15, we see that the condition number of $\mathbf{A}_N\mathbf{P}_N^{-1}$ is much smaller than that of $\mathbf{A}_N$. Moreover, the condition number of $\mathbf{A}_N\mathbf{P}_N^{-1}$ increases sublinearly while condition number of $\mathbf{A}_N$ increases linearly with respect to $\max\{\eta_x, \eta_y\}$. It means the splitting preconditioning technique improves the condition number of $\mathbf{A}_N\mathbf{P}_N^{-1}$ even if the theoretical assumptions are not met. On the other hand, unlike the condition number of $\mathbf{A}_N\mathbf{P}_N^{-1}$ that stays almost unchanged in Table 10, it keeps increasing as $\max\{\eta_x, \eta_y\}$ increases in Table 15. It means that when the theoretical assumptions are not satisfied there indeed exist some cases where the preconditioned matrix has a condition number dependent on the discretization parameters, which implies the sharpness of our theoretical results.

TABLE 11
Results of different solvers when $N = 2^4$ for Example 5.3.

		GMRES-**SP**		GMRES-**P**(5)		GMRES-**B**(15)		GMRES-**C**	
(α,β)	$M+1$	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	7.0	1.69s	22.0	7.77s	24.0	4.28s	16.0	1.11s
	2^8	7.0	6.84s	27.0	35.37s	35.5	24.47s	18.0	5.48s
	2^9	7.0	37.18s	35.0	266.03s	52.9	216.49s	20.0	39.22s
(1.1,1.6)	2^7	7.0	1.65s	28.5	9.81s	25.0	4.33s	20.0	1.32s
	2^8	7.0	6.83s	37.0	48.16s	37.0	25.27s	23.9	7.30s
	2^9	7.0	37.31s	49.0	392.18s	54.4	223.10s	27.4	57.94s
(1.1,1.9)	2^7	7.0	1.65s	39.0	13.33s	25.4	4.33s	24.9	1.71s
	2^8	7.0	6.88s	55.3	74.38s	36.6	24.96s	30.8	9.87s
	2^9	7.0	36.93s	80.1	713.21s	52.8	214.89s	38.1	92.64s
(1.3,1.3)	2^7	7.0	1.66s	22.0	7.94s	30.0	4.84s	18.0	1.20s
	2^8	7.0	6.90s	29.0	37.60s	45.9	30.30s	20.0	6.04s
	2^9	7.0	37.16s	39.0	293.45s	71.0	316.67s	23.0	46.20s
(1.6,1.6)	2^7	8.0	1.83s	25.0	8.43s	51.0	7.59s	20.9	1.40s
	2^8	7.5	7.23s	33.0	42.09s	86.6	60.99s	24.1	7.38s
	2^9	7.0	36.84s	44.0	335.58s	148.2	1093.28s	28.4	60.81s
(1.9,1.9)	2^7	9.0	2.01s	29.0	9.73s	91.3	16.51s	24.0	1.66s
	2^8	8.0	7.66s	38.0	48.72s	172.4	184.20s	28.8	9.09s
	2^9	8.0	41.36s	51.0	397.97s	341.3	3914.50s	35.2	82.33s

TABLE 12
Results of different solvers when $N = 2^4$ for Example 5.3.

		MGM-**TS**		GMRES-$\mathbf{S_1}$		GMRES-$\mathbf{S_2}$		GMRES-**I**	
(α,β)	$M+1$	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^7	12.6	2.82s	42.2	6.78s	42.2	6.80s	86.6	9.43s
	2^8	14.1	11.19s	56.1	39.94s	56.1	39.95s	138.3	90.12s
	2^9	15.4	80.23s	69.3	321.08s	69.3	322.61s	219.1	1811.68s
(1.1,1.6)	2^7	29.4	5.96s	52.4	8.71s	52.3	8.74s	160.1	28.36s
	2^8	39.0	25.57s	70.6	52.49s	70.5	52.86s	282.5	327.40s
	2^9	50.7	229.45s	91.1	481.16s	91.0	486.12s	654.8	6618.33s
(1.1,1.9)	2^7	36.4	7.26s	73.4	13.36s	73.4	13.38s	344.4	95.52s
	2^8	59.1	37.84s	96.4	79.92s	96.4	79.48s	908.0	1105.24s
	2^9	94.7	432.90s	125.3	804.72s	125.3	802.03s	2540.3	26716.07s
(1.3,1.3)	2^7	7.5	1.82s	31.1	5.38s	31.1	4.83s	91.0	10.10s
	2^8	7.7	6.90s	38.7	26.69s	38.7	25.72s	147.9	99.78s
	2^9	8.4	48.39s	45.1	176.70s	45.1	176.83s	237.0	2052.53s
(1.6,1.6)	2^7	7.5	1.82s	22.0	3.45s	22.0	3.44s	153.4	26.19s
	2^8	7.5	6.73s	25.5	17.31s	25.5	16.88s	275.2	310.05s
	2^9	7.5	43.84s	28.4	101.60s	28.4	100.88s	699.8	6858.06s
(1.9,1.9)	2^7	14.1	3.11s	19.0	3.01s	19.0	3.06s	268.8	75.41s
	2^8	14.1	10.81s	21.0	14.34s	21.0	14.33s	820.6	941.24s
	2^9	14.1	76.05s	23.0	80.84s	23.0	80.06s	3048.1	33748.48s

TABLE 13
Results of different solvers when $M + 1 = 2^9$ for Example 5.3.

		GMRES-**SP**		GMRES-**P**(5)		GMRES-**B**(15)		GMRES-**C**	
(α,β)	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^1	7.0	4.99s	35.0	35.86s	53.0	28.44s	20.0	4.94s
	2^2	7.0	9.53s	35.0	69.53s	53.0	55.97s	20.0	9.82s
	2^3	7.0	18.89s	35.0	142.44s	53.0	112.05s	20.0	19.24s
(1.1,1.6)	2^1	7.0	4.85s	49.0	51.76s	54.5	28.21s	27.0	7.03s
	2^2	7.0	9.53s	49.0	97.57s	54.5	56.32s	27.3	14.27s
	2^3	7.0	18.82s	49.0	196.35s	54.5	113.28s	27.5	28.87s
(1.1,1.9)	2^1	7.0	4.83s	80.5	91.04s	52.5	27.00s	38.0	11.36s
	2^2	7.0	9.48s	80.3	175.16s	52.8	55.64s	38.0	22.78s
	2^3	7.0	18.74s	80.1	365.13s	52.8	113.25s	38.0	45.60s
(1.3,1.3)	2^1	7.0	4.83s	39.0	38.09s	71.0	43.04s	23.0	5.68s
	2^2	7.0	9.77s	39.0	78.09s	71.0	82.52s	23.0	11.41s
	2^3	7.0	19.48s	39.0	149.48s	70.9	169.62s	23.0	22.85s
(1.6,1.6)	2^1	7.0	4.80s	44.0	44.33s	147.5	130.54s	28.0	7.35s
	2^2	7.0	9.62s	44.0	89.86s	148.0	273.77s	28.0	14.80s
	2^3	7.0	18.50s	44.0	176.85s	147.9	550.39s	28.0	29.59s
(1.9,1.9)	2^1	8.0	5.28s	51.0	52.40s	339.0	494.57s	35.0	10.09s
	2^2	8.0	10.61s	51.0	104.86s	340.8	931.38s	35.0	20.20s
	2^3	8.0	21.01s	51.0	203.38s	341.0	1828.63s	35.0	40.47s

Table 14
Results of different solvers when $M+1=2^9$ for Example 5.3.

		MGM-**TS**		GMRES-$\mathbf{S_1}$		GMRES-$\mathbf{S_2}$		GMRES-**I**	
(α,β)	N	Iter.	CPU	Iter.	CPU	Iter.	CPU	Iter.	CPU
(1.1,1.3)	2^1	8.5	9.37s	69.5	46.35s	69.5	40.58s	218.5	228.55s
	2^2	12.5	19.74s	69.3	93.04s	69.3	80.61s	218.5	482.84s
	2^3	14.7	37.32s	69.3	182.57s	69.3	161.77s	218.8	911.79s
(1.1,1.6)	2^1	28.0	28.75s	91.0	68.75s	91.0	60.82s	653.0	849.59s
	2^2	40.8	55.86s	91.0	133.84s	91.0	121.70s	654.0	1704.11s
	2^3	47.4	112.43s	90.8	262.75s	91.1	243.91s	654.5	3427.29s
(1.1,1.9)	2^1	52.5	51.04s	125.5	112.52s	125.5	101.38s	2543.5	3458.69s
	2^2	76.3	101.20s	125.3	202.86s	125.3	202.24s	2543.5	7081.14s
	2^3	88.4	200.84s	125.3	404.86s	125.3	405.29s	2542.1	13824.80
(1.3,1.3)	2^1	4.5	5.67s	44.5	21.96s	45.0	22.36s	236.5	266.97s
	2^2	6.8	11.33s	44.5	43.87s	44.5	43.80s	236.5	533.87s
	2^3	7.9	22.65s	45.0	88.90s	45.0	88.81s	236.9	1065.60s
(1.6,1.6)	2^1	4.0	5.20s	28.5	12.85s	28.5	12.85s	703.0	881.22s
	2^2	6.0	10.36s	28.3	25.42s	28.3	25.37s	700.8	1806.59s
	2^3	7.0	20.74s	28.4	50.91s	28.4	50.94s	700.3	3446.08s
(1.9,1.9)	2^1	7.5	8.58s	23.0	10.24s	23.0	10.21s	3052.0	4294.97s
	2^2	11.3	17.13s	23.0	20.37s	23.0	20.38s	3050.8	8149.41s
	2^3	13.1	34.47s	23.0	40.69s	23.0	40.66s	3049.0	16461.13s

Table 15
Condition numbers of $\mathbf{A}_N$ and the preconditioned matrix $\mathbf{A}_N\mathbf{P}_N^{-1}$ by the splitting preconditioner in Example 5.3 for different values of η_x and η_y with $\eta_x = (M+1)^\alpha/(2^\alpha N)$ and $\eta_y = (M+1)^\beta/(2^\beta N)$.

$(\alpha,\beta)=(1.1,1.3)$				$(\alpha,\beta)=(1.1,1.6)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
9.85	14.93	37.76	1.50	9.85	27.86	66.45	1.54
21.11	36.76	96.92	1.91	21.11	84.45	205.98	1.98
45.25	90.51	243.80	3.02	45.25	256.00	624.91	2.49
$(\alpha,\beta)=(1.1,1.9)$				$(\alpha,\beta)=(1.3,1.3)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
9.85	51.98	128.72	1.60	14.93	14.93	42.63	1.61
21.11	194.01	492.36	1.89	36.76	36.76	116.16	2.49
45.25	724.08	1.85E3	2.33	90.51	90.51	302.37	4.04
$(\alpha,\beta)=(1.6,1.6)$				$(\alpha,\beta)=(1.9,1.9)$			
η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$	η_x	η_y	$\mathbf{A}_N$	$\mathbf{A}_N\mathbf{P}_N^{-1}$
27.86	27.86	82.97	1.75	51.98	51.98	169.30	1.86
84.45	84.45	277.97	2.83	194.01	194.01	693.47	3.06
256.00	256.00	885.23	4.39	724.08	724.08	2.71E3	5.49

6. Concluding remarks. In this paper, we have considered discretized linear systems arising from one-dimensional and two-dimensional fractional diffusion equations. The coefficient matrix is a sum of an identity matrix and a diagonal-times-Toeplitz matrices. In the two-dimensional case, the involved Toeplitz matrices are two-level. Discretization parameter–independent preconditioning techniques for such matrices have not been studied yet. The main contribution of this paper is to propose a splitting preconditioner for such linear systems so that the Krylov subspace method for the preconditioned linear system has a fast convergence and low cost of computational times. Theoretically, we show that singular values of the preconditioned matrix are uniformly bounded above and below by constants independent of discretization parameters under certain conditions on diffusion coefficients. Numerical experiments support our theoretical analysis and demonstrate efficiency of the splitting preconditioner. In future research work, we will extend the preconditioning strategy to time-fractional partial differential equations and take into account the inexact implementation in the spectra analysis.

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