Stability and convergence of finite difference schemes for a class of time-fractional sub-diffusion equations based on certain superconvergence ☆

Guang-Hua Gao[a], Hai-Wei Sun[b,*], Zhi-Zhong Sun[c]

[a]*College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, P.R. China*
[b]*Department of Mathematics, University of Macau, Macao*
[c]*Department of Mathematics, Southeast University, Nanjing 210096, P.R. China*

Abstract

This paper is devoted to the construction and analysis of finite difference methods for solving a class of time-fractional subdiffusion equations. Based on the certain superconvergence at some particular points of the fractional derivative by the traditional first-order Grünwald-Letnikov formula, some effective finite difference schemes are derived. The obtained schemes can achieve the global second-order numerical accuracy in time, which is independent of the values of anomalous diffusion exponent α $(0 < \alpha < 1)$ in the governing equation. The spatial second-order scheme and the fourth-order compact scheme, respectively, are established for the one-dimensional problem along with the strict analysis on unconditional stability and convergence of these schemes by the discrete energy method. Furthermore, the extension to the two-dimensional case is also considered. Numerical experiments support the correctness of the theoretical analysis and effectiveness of the new developed difference schemes.

Keywords: Time-fractional sub-diffusion equations; Grünwald-Letnikov formula; Finite difference scheme; Stability; Convergence.

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1. Introduction

In the recent few decades, the remarkable applications of fractional calculus in diverse engineering fields have been gradually realized and meanwhile, the discussion on the related fractional differential equations becomes a hot topic of many scholars. To seek the exact solutions to these differential equations is not an easy job in spite of some research results

☆The research was supported by the research grant 11271068, 11326225, 11401319 from National Natural Science Foundation of China, BK20130860 from Natural Science Youth Foundation of Jiangsu Province, NY213051 from the Scientific Research Foundation of Nanjing University of Posts and Telecommunications, 105/2012/A3 from FDCT of Macao, and MYRG102(Y2-L3)-FST13-SHW from University of Macau.

*Corresponding author

Email addresses: gaoguanghua1107@163.com (Guang-Hua Gao), HSun@umac.mo (Hai-Wei Sun), zzsun@seu.edu.cn (Zhi-Zhong Sun)

around the world on the subject [1, 2, 3, 4]. Effective and simple numerical methods for solving these equations tend to be favored in practical computations, for instance, readers can refer to the works [5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

Anomalous diffusion equations are often used to describe the transport dynamics in various complex systems where Gaussian statistics are no longer followed and the Fick's second law fails to describe the related transport behaviors. Anomalous diffusion in the presence of an external velocity or force field has been modelled in numerous way, one of which is given in terms of continuous time random walk (CTRW) models. Based on the CTRW models, a generalized diffusion equation of fractional order is derived in [15]. Until now, two forms of this kind of equations often have appeared: one is written as [16, 17, 18]

$$ {}_0^C\mathcal{D}_t^{\alpha}u(x,t) = u_{xx}(x,t) + f(x,t),\ (x,t) \in (a,b) \times (0,T], \tag{1.1} $$

where $0 < \alpha < 1$ and ${}_0^C\mathcal{D}_t^{\alpha}$ is the α-th order Caputo time-fractional operator defined by

$$ {}_0^C\mathcal{D}_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\partial_s u(x,s)\,\mathrm{d}s, $$

and we call it the time Caputo-type subdiffusion case; the other takes the form of [19, 20, 21, 23, 26]

$$ u_t(x,t) = {}_0^{RL}\mathcal{D}_t^{1-\alpha}u_{xx}(x,t) + g(x,t),\ (x,t) \in (a,b) \times (0,T], \tag{1.2} $$

where $0 < \alpha < 1$ and ${}_0^{RL}\mathcal{D}_t^{\alpha}$ is the α-th order Riemann-Liouville fractional operator defined by

$$ {}_0^{RL}\mathcal{D}_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha}u(x,s)\,\mathrm{d}s, $$

and we call it the time Riemann-Liouville-type subdiffusion equation. These two forms are equivalent under some regularity assumption for $u(x,t)$ in time and $g(x,t) = {}_0^{RL}\mathcal{D}_t^{1-\alpha}f(x,t)$; see, for details, [27].

Traditionally, the direct difference approximation for the time-fractional derivative covers two ways: the $L1$ formula and the Grünwald-Letnikov formula. The former is based on the piecewise linear interpolating approximation with respect to t for the integrand $u(x,t)$ inside the integral in the Caputo fractional derivative sense, while the latter is often used to handle the Riemann-Liouville time-fractional derivative. For the α-th $(0 < \alpha < 1)$ fractional derivative, the numerical accuracy of $L1$ formula is proved to be $2-\alpha$ [16, 17], which is less than two, and that of the Grünwald-Letnikov formula depends on the choice of generating function of coefficients. The common generating function of coefficients in this formula is chosen to be $(1-z)^{\alpha}$, and only the first-order accuracy is attained [20, 21, 22, 23, 24, 25].

Recently, the great efforts to enhance the numerical accuracy of approximating time-fractional derivatives have been made. Cao and Xu [28] started from the equivalent Volterra integral form of the original time Caputo-type fractional differential equations to design the high order numerical scheme in time. The computation of coefficients in related numerical methods is quite complex and expensive. Gao, Sun and Zhang [29] presented a modified $L1$ numerical differentiation formula to directly discretize the Caputo time-fractional derivative and higher-order numerical accuracy seems to be realized, where the strict convergence analysis for the corresponding difference scheme has not been available. Wang and Vong [30] established some difference schemes with the second-order accuracy in time for solving the time-fractional subdiffusion and diffusion-wave equations by mean of weighed Grünwald-Letnikov formula. The similar techniques were used to deal with the problem with Neumann boundary conditions in [31]. Ding and Li [32] proposed a second-order difference approximation for the Riemann-Liouville time-fractional derivative by the Grünwald-Letnikov formula with the generating function of coefficients as $(3/2 - 2z + z^2/2)^\alpha$. The obtained coefficients were more complex than the usual first-order Grünwald-Letnikov formula. Here, we shall show the alternative way to design the high-order finite difference scheme using the simple first-order Grünwald-Letnikov formula to approximate the Riemann-Liouville time-fractional derivative. Zhao and Deng [34] investigated a series of high order pseudo-compact schemes for space fractional diffusion equations based on the superconvergent approximations for fractional derivatives.

The key point of this approach is based on the significant work by Nasir et al. [33], where the superconvergent points of the first-order Grünwald-Letnikov formula to approximate the Riemann-Liouville time-fractional derivative are exactly pinpointed. Namely, the standard first-order Grünwald-Letnikov formula or shifted first-order Grünwald-Letnikov formula to approximate the fractional derivative value at current point is only first-order accurate, whereas one-order higher numerical accuracy at some shifted positions can be obtained. To our knowledge, the technique exploited such superconvergence to directly improve the numerical accuracy of approximating the time-fractional derivative has not appeared in the literature, except the very recent work by Dimitrov in [35] for the one-dimensional time Caputo-type subdiffusion case. In order to clarify the success of this idea, we begin with the one-dimensional problem and construct a second-order accurate difference scheme both in time and space for solving the time-fractional sub-diffusion equation. Then the spatial fourth-order compact scheme is also established along the similar route. Furthermore, the extension to the two-dimensional case is also taken into account.

The outline of this paper is as follows. In section 2, a second-order difference scheme both in time and space is derived by considering the governing equation at the superconvergent point of the standard first-order Grünwald-Letnikov formula for the Riemann-Liouville derivative. The unconditional stability and convergence of the second-order scheme are proved in section 3 by the discrete energy method. In section 4, the compact difference scheme with the convergence of second order in time and fourth order in space is constructed. The stability and convergence are also given. Section 5 is devoted to the discussion of the two-dimensional case. Numerical examples are included in section 6 to verify the efficiency of the proposed schemes. A brief conclusion ends this work finally.

2. A second-order finite difference scheme

In this paper, we begin with the study on the following one-dimensional time-fractional sub-diffusion equations

$$u_t(x,t) = {}_0^{RL}\mathcal{D}_t^{\alpha} u_{xx}(x,t) + f(x,t), \quad x \in (0,L),\ t \in (0,T], \tag{2.1}$$

$$u(x,0) = 0, \quad x \in [0,L], \tag{2.2}$$

$$u(0,t) = \phi_1(t), \quad u(L,t) = \phi_2(t) \quad 0 < t \leqslant T, \tag{2.3}$$

where $L > 0$, $0 < \alpha < 1$ are constants, $f(x,t)$, $\phi_1(t)$, $\phi_2(t)$ are all given and smooth enough functions.

For numerical approximation, firstly, a mesh partition is introduced. For any two positive integers M and N, denote $x_i = ih (i = 0,1,\ldots,M)$, $t_k = k\tau (k = 0,1,\ldots,N)$ with $h = L/M$ and $\tau = T/N$. Let $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$ and $\Omega_\tau = \{t_k \mid 0 \leqslant k \leqslant N\}$, then the domain $[0,L] \times [0,T]$ is covered by $\Omega_h \times \Omega_\tau$. Let $\mathcal{V}_h = \{u \mid u = (u_0, u_1, \ldots, u_M), u_0 = u_M = 0\}$ be grid function space defined on Ω_h and $t_{k-\alpha/2} = t_k - \alpha\tau/2$ $(1 \leqslant k \leqslant N)$.

In addition, some notations are brought here for any mesh function $u \in \mathcal{V}_h$ as

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1})\ (1 \leqslant i \leqslant M), \quad \delta_x^2 u_i = \frac{1}{h}\Big(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}\Big)\ (1 \leqslant i \leqslant M-1).$$

Introduce the discrete inner products and the corresponding norms for any $u, v \in \mathcal{V}_h$ as follows:

$$(u,v) = h\sum_{i=1}^{M-1} u_i v_i,\ (\delta_x u, \delta_x v) = h\sum_{i=1}^{M} \Big(\delta_x u_{i-\frac{1}{2}}\Big)\Big(\delta_x v_{i-\frac{1}{2}}\Big),$$

$$\|u\| = \sqrt{(u,u)}, \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)}.$$

If $w = \{w^k \mid 0 \leqslant k \leqslant N\}$ is a grid function defined on Ω_τ, denote

$$\delta_t w^{k-1/2} = \frac{1}{\tau}(w^k - w^{k-1})\ (1 \leqslant k \leqslant N),$$

$$D_{\hat{t}} w^k = \frac{1}{2\tau}\Big[(3-\alpha)w^k - (4-2\alpha)w^{k-1} + (1-\alpha)w^{k-2}\Big]\ (2 \leqslant k \leqslant N).$$

Secondly, for the sake of numerical computation on the Riemann-Liouville fractional derivative of function $g(t)$, denote

$$\delta_{\tau,p}^{\alpha} g(t) = \tau^{-\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)} g\Big(t - (k-p)\tau\Big),$$

where

$$w_k^{(\alpha)} = (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}, k \geqslant 0,$$

are the coefficients of the power series of the generating function $(1-z)^\alpha$, namely, $(1-z)^\alpha = \sum\limits_{k=0}^{\infty} w_k^{(\alpha)} z^k$ and the coefficients satisfy the following recursive formulae

$$w_0^{(\alpha)} = 1, \quad w_k^{(\alpha)} = \Big(1 - \frac{\alpha+1}{k}\Big) w_{k-1}^{(\alpha)}, \ k \geqslant 1.$$

It is easy to know that [21]

$$w_0^{(\alpha)} = 1 > 0, \ w_k^{(\alpha)} < 0, k = 1, 2, \dots, \ \text{and} \ \sum_{k=1}^{\infty} w_k^{(\alpha)} = -1.$$

Next, two lemmas are listed which will be the foundation of the construction of the aftermentioned finite difference schemes.

Lemma 1. *[34] Suppose $0 < \alpha < 1$, $g(t) \in C^2[0, +\infty)$, $D^3 g(t) \in L^1[0, +\infty)$ and $D^k g(0) = 0, k = 0, 1, 2$. Then for any integer p and a real parameter β, there holds*

$${}_0^{RL}\mathcal{D}_t^\alpha g(t + \beta\tau) - \delta_{\tau,p}^\alpha g(t) = \Big[\beta - \big(p - \frac{\alpha}{2}\big)\Big] {}_0^{RL}\mathcal{D}_t^{1+\alpha} g(t) \cdot \tau + \mathcal{O}(\tau^2)$$

for $t + \beta\tau \in [0, +\infty)$.

Remark: The lemma in [34] is proved for the space function defined on a bounded space domain using a Fourier transform and the derivative conditions at two boundaries are needed. Here, we apply this result to the time function and correspondingly, the derivative conditions at the initial time are needed.

The lemma reveals that only the first-order accuracy is obtained for general parameters p and β if the operator $\delta_{\tau,p}^\alpha g(t)$ is used to numerically approximate the α-th order Riemann-Liouville fractional derivative of function $g(t + \beta\tau)$. Nevertheless, one can surprisingly discover that some certain superconvergence occurred if $\beta - (p - \alpha/2) = 0$. Particularly, when $p = 0$ and $\beta = -\alpha/2$, we have

$${}_0^{RL}\mathcal{D}_t^\alpha g\Big(t - \frac{\alpha}{2}\tau\Big) - \delta_{\tau,0}^\alpha g(t) = \mathcal{O}(\tau^2), \tag{2.4}$$

provided that the conditions of Lemma 1 hold. It is one of the important and fundamental starting points of our work here.

The subsequent numerical treatment of $u_t(x, t)$ needs the next lemma.

Lemma 2. *Suppose $v(t) \in C^3[0, T]$. It holds*

$$\begin{aligned} D_{\hat{t}} v(t_k) &= \frac{1}{2\tau}[(3-\alpha)v(t_k) - (4-2\alpha)v(t_{k-1}) + (1-\alpha)v(t_{k-2})] \\ &= \frac{\mathrm{d}v}{\mathrm{d}t}\Big(t_{k-\frac{\alpha}{2}}\Big) + \mathcal{O}(\tau^2), \quad k \geqslant 2. \end{aligned}$$

Proof. The lemma can be proved by directly taking the Taylor's expansion of function $v(t)$ for $t = t_k, t_{k-1}$ and t_{k-2} at the point $t_{k-\frac{\alpha}{2}}$, respectively. The details are omitted here. □

Now we turn to consider the derivation of an effective finite difference scheme for solving the problem (2.1)–(2.3). Suppose $u(x,t) \in C^{(4,3)}([0,L]\times[0,T])$ and $\partial^k u(x,0)/\partial t^k = 0$ for $k = 0,1,2$. The derivative conditions at $t = 0$ is reasonable due to the work in [35]. Denote $U_i^k = u(x_i,t_k)$, $0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N$. Considering Eq. (2.1) at points $(x_i, t_{k-\alpha/2})$, we get

$$u_t(x_i,t_{k-\frac{\alpha}{2}}) = {}_0^{RL}\mathcal{D}_t^{\alpha} u_{xx}(x_i,t_{k-\frac{\alpha}{2}}) + f(x_i,t_{k-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N. \tag{2.5}$$

By Lemma 2, we have

$$u_t(x_i,t_{k-\frac{\alpha}{2}}) = D_{\hat{t}} U_i^k + \mathcal{O}(\tau^2). \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N. \tag{2.6}$$

By (2.4) and the Taylor's expansion, it follows

$$\begin{aligned}
&{}_0^{RL}\mathcal{D}_t^{\alpha} u_{xx}(x_i,t_{k-\frac{\alpha}{2}}) = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)} u_{xx}(x_i,t_{k-l}) + \mathcal{O}(\tau^2)\\
&= \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\Big\{\delta_x^2 U_i^{k-l} - \frac{h^2}{6}\int_0^1 \Big[\frac{\partial^4 u}{\partial x^4}(x_i+\lambda h, t_{k-l}) + \frac{\partial^4 u}{\partial x^4}(x_i-\lambda h, t_{k-l})\Big](1-\lambda)^3\,\mathrm{d}\lambda\Big\}\\
&\quad + \mathcal{O}(\tau^2)\\
&= \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 U_i^{k-l} + \mathcal{O}(\tau^2+h^2),\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N.
\end{aligned} \tag{2.7}$$

The substitution of (2.6) and (2.7) into (2.5) yields

$$D_{\hat{t}} U_i^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 U_i^{k-l} + f(x_i,t_{k-\frac{\alpha}{2}}) + R_i^k,\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N, \tag{2.8}$$

where there is a $c_1 > 0$ such that

$$|R_i^k| \leqslant c_1(\tau^2+h^2), \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N. \tag{2.9}$$

For $k = 1$, a simple difference scheme can be used to numerically compute the values of $u(x,t)$ at $t = t_1$. Here we take

$$\delta_t U_i^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\delta_x^2 U_i^{1-l} + f(x_i,t_{1-\frac{\alpha}{2}}) + R_i^1,\ 1 \leqslant i \leqslant M-1, \tag{2.10}$$

where, by Taylor's expansion and (2.4), there exists a positive constant c_2 satisfying

$$|R_i^1| \leqslant c_2(\tau+h^2), \quad 1 \leqslant i \leqslant M-1. \tag{2.11}$$

Neglecting small terms in Eqs. (2.8) and (2.10) and noticing the initial-boundary value conditions (2.2)–(2.3), an effective finite difference scheme for solving the problem (2.1)–(2.3) can be proposed in the form of

$$D_{\hat{t}}u_i^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 u_i^{k-l} + f(x_i, t_{k-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N, \tag{2.12}$$

$$\delta_t u_i^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\delta_x^2 u_i^{1-l} + f(x_i, t_{1-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1, \tag{2.13}$$

$$u_i^0 = 0, \quad 0 \leqslant i \leqslant M, \tag{2.14}$$

$$u_0^k = \phi_1(t_k), \quad u_M^k = \phi_2(t_k),\ 1 \leqslant k \leqslant N. \tag{2.15}$$

The finite difference scheme (2.12)–(2.15) is a linear algebraic system with respect to the unknowns $\{u_i^k\,|0 \leqslant i \leqslant M\}$ on the time level $t = t_k$ $(1 \leqslant k \leqslant N)$ and the coefficient matrix is tri-diagonal and strictly diagonally dominant. Hence, the solution of (2.12)–(2.15) is unique and the Thomas algorithm can be used.

3. Stability and Convergence

In this section, the stability and convergence of finite difference scheme (2.12)–(2.15) will be analyzed. For this, two lemmas are given at first.

Lemma 3. *For any mesh function* $\{v^k|k = 0, 1, \ldots, N\}$, *it holds*

$$\sum_{k=0}^{n}\sum_{l=0}^{k} w_l^{(\alpha)} v^{k-l} v^k \geqslant 0, \quad 0 \leqslant n \leqslant N.$$

Proof. It is enough to prove that the toeplitz matrix

$$W = \begin{pmatrix} w_0^{(\alpha)} & \frac{w_1^{(\alpha)}}{2} & \cdots & \frac{w_{n-1}^{(\alpha)}}{2} & \frac{w_n^{(\alpha)}}{2} \\ \frac{w_1^{(\alpha)}}{2} & w_0^{(\alpha)} & \frac{w_1^{(\alpha)}}{2} & \cdots & \frac{w_{n-1}^{(\alpha)}}{2} \\ \vdots & \frac{w_1^{(\alpha)}}{2} & w_0^{(\alpha)} & \ddots & \vdots \\ \frac{w_{n-1}^{(\alpha)}}{2} & \vdots & \ddots & \ddots & \frac{w_1^{(\alpha)}}{2} \\ \frac{w_n^{(\alpha)}}{2} & \frac{w_{n-1}^{(\alpha)}}{2} & \cdots & \frac{w_1^{(\alpha)}}{2} & w_0^{(\alpha)} \end{pmatrix}$$

is positive definite.

Denote $\mathbf{i}=\sqrt{-1}$. The generating function of matrix W is

$$\begin{aligned}
f(x) &= w_0^{(\alpha)} + \frac{1}{2}\sum_{k=1}^{\infty} w_k^{(\alpha)} \mathrm{e}^{\mathbf{i}kx} + \frac{1}{2}\sum_{k=1}^{\infty} w_k^{(\alpha)} \mathrm{e}^{-\mathbf{i}kx} \\
&= \frac{1}{2}\sum_{k=0}^{\infty} w_k^{(\alpha)} \mathrm{e}^{\mathbf{i}kx} + \frac{1}{2}\sum_{k=0}^{\infty} w_k^{(\alpha)} \mathrm{e}^{-\mathbf{i}kx} \\
&= \frac{1}{2}[(1-\mathrm{e}^{\mathbf{i}x})^\alpha + (1-\mathrm{e}^{-\mathbf{i}x})^\alpha] \\
&= \frac{1}{2}\left(2\sin\frac{x}{2}\right)^\alpha \left[\mathrm{e}^{\mathbf{i}\frac{\alpha}{2}(x-\pi)} + \mathrm{e}^{\mathbf{i}\frac{\alpha}{2}(\pi-x)}\right] \\
&= \left(2\sin\frac{x}{2}\right)^\alpha \cos\left[\frac{\alpha}{2}(\pi-x)\right], \quad x\in[-\pi,\pi].
\end{aligned}$$

It is easy to find that $f(x)$ is a real and even function, thus, only the case of $f(x)$ on $x\in[0,\pi]$ need be judged. Apparently, when $x\in[0,\pi]$, the values of $f(x)$ are nonnegative. By the Grenander-Szegö theorem [36], the conclusion that the toeplitz matrix W is positive definite can be drawn. The proof is completed. □

Lemma 4. *For any mesh function $\{v^k \mid k=0,1,\ldots,N\}$ and $0<\alpha<1$, we have*

$$(D_{\hat{t}}v^k, v^k) \geqslant \frac{1}{4\tau}(E^k - E^{k-1}), \quad k\geqslant 2, \tag{3.1}$$

with

$$E^k = (3-\alpha)\|v^k\|^2 - (1-\alpha)\|v^{k-1}\|^2 + 2\|v^k - v^{k-1}\|^2, \quad k\geqslant 1.$$

In addition, it holds

$$E^k \geqslant \|v^k\|^2, \quad k\geqslant 1. \tag{3.2}$$

Proof. It is apparent that the operator $D_{\hat{t}}v^k$ can be rewritten as

$$D_{\hat{t}}v^k = (2-\alpha)\frac{v^k - v^{k-1}}{\tau} - (1-\alpha)\frac{v^k - v^{k-2}}{2\tau},$$

and noticing the identity $(a-b)a=[(a^2-b^2)+(a-b)^2]/2$, we have

$$\begin{aligned}(D_{\widehat{t}}v^k,v^k)&=(2-\alpha)\Big(\frac{v^k-v^{k-1}}{\tau},v^k\Big)-(1-\alpha)\Big(\frac{v^k-v^{k-2}}{2\tau},v^k\Big)\\&=(2-\alpha)\frac{1}{2\tau}(\|v^k\|^2-\|v^{k-1}\|^2+\|v^k-v^{k-1}\|^2)\\&\quad-(1-\alpha)\frac{1}{4\tau}(\|v^k\|^2-\|v^{k-2}\|^2+\|v^k-v^{k-2}\|^2)\\&\geqslant\frac{2-\alpha}{2}\frac{1}{\tau}(\|v^k\|^2-\|v^{k-1}\|^2)-\frac{1-\alpha}{2}\frac{1}{2\tau}(\|v^k\|^2-\|v^{k-2}\|^2)\\&\quad+\frac{2-\alpha}{2}\frac{1}{\tau}\|v^k-v^{k-1}\|^2-\frac{1-\alpha}{2}\frac{1}{\tau}(\|v^k-v^{k-1}\|^2+\|v^{k-1}-v^{k-2}\|^2)\\&=\frac{2-\alpha}{2}\frac{1}{\tau}(\|v^k\|^2-\|v^{k-1}\|^2)-\frac{1-\alpha}{2}\frac{1}{2\tau}(\|v^k\|^2-\|v^{k-2}\|^2)\\&\quad+\frac{1}{2\tau}\|v^k-v^{k-1}\|^2-\frac{1-\alpha}{2\tau}\|v^{k-1}-v^{k-2}\|^2\\&\geqslant\frac{2-\alpha}{2}\frac{1}{\tau}(\|v^k\|^2-\|v^{k-1}\|^2)-\frac{1-\alpha}{2}\frac{1}{2\tau}(\|v^k\|^2-\|v^{k-2}\|^2)\\&\quad+\frac{1}{2\tau}(\|v^k-v^{k-1}\|^2-\|v^{k-1}-v^{k-2}\|^2)\\&=\frac{1}{4\tau}(E^k-E^{k-1}),\quad k\geqslant 2,\end{aligned}$$

Which is just the inequality (3.1). Further, a direct calculation proceeds to show

$$\begin{aligned}E^k&=(5-\alpha)\|v^k\|^2+(1+\alpha)\|v^{k-1}\|^2-4(v^k,v^{k-1})\\&=\Big(5-\alpha-\frac{4}{1+\alpha}\Big)\|v^k\|^2+\Big\|\frac{2}{\sqrt{1+\alpha}}\,v^k-\sqrt{1+\alpha}\,v^{k-1}\Big\|^2\\&\geqslant\Big(5-\alpha-\frac{4}{1+\alpha}\Big)\|v^k\|^2\geqslant\|v^k\|^2,\end{aligned}$$

where the last inequality is based on the fact that the function $q(\alpha)=5-\alpha-\frac{4}{1+\alpha}$ is monotone increasing for $\alpha\in(0,1)$ and $q(0)=1$. The proof ends. □

Secondly, a prior estimation theorem is illustrated.

Theorem 1. *Suppose* $\{v_i^k\mid 0\leqslant i\leqslant M, 0\leqslant k\leqslant N\}$ *is the solution of*

$$D_{\widehat{t}}v_i^k=\tau^{-\alpha}\sum_{l=0}^{k}w_l^{(\alpha)}\delta_x^2v_i^{k-l}+g_i^k,\ 1\leqslant i\leqslant M-1,\ 2\leqslant k\leqslant N,\tag{3.3}$$

$$\delta_t v_i^{\frac{1}{2}}=\tau^{-\alpha}\sum_{l=0}^{1}w_l^{(\alpha)}\delta_x^2v_i^{1-l}+g_i^1,\ 1\leqslant i\leqslant M-1,\tag{3.4}$$

$$v_i^0=\varphi(x_i),\quad 0\leqslant i\leqslant M,\tag{3.5}$$

$$v_0^k=v_M^k=0,\ 1\leqslant k\leqslant N.\tag{3.6}$$

Then if $\tau \leqslant 1/2$, it holds

$$\|v^n\|^2 \leqslant \Big[(5-\alpha)\|v^0\|^2 + \Big(4+(2-\alpha)\alpha^2\Big)\tau^{1-\alpha}\|\delta_x v^0\|^2 + 4(3-\alpha)\|\tau g^1\|^2 + 4\tau\sum_{k=2}^{n}\|g^k\|^2\Big]\exp(2n\tau), \quad n \geqslant 1.$$

Proof. Taking the inner product of Eq. (3.4) with v^1 on the both sides, noticing (3.6) and using the summation formula by parts, we get

$$\begin{aligned}\Big(\delta_t v^{\frac{1}{2}}, v^1\Big) &= \frac{1}{2\tau}(\|v^1\|^2 - \|v^0\|^2) + \frac{\tau}{2}\|\delta_t v^{\frac{1}{2}}\|^2 \\ &= \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}(\delta_x^2 v^{1-l}, v^1) + (g^1, v^1) \\ &= -\tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}(\delta_x v^{1-l}, \delta_x v^1) + (g^1, v^1),\end{aligned} \tag{3.7}$$

then it follows that

$$\begin{aligned}&\frac{1}{2\tau}(\|v^1\|^2 - \|v^0\|^2) + \frac{\tau}{2}\|\delta_t v^{\frac{1}{2}}\|^2 = -\tau^{-\alpha}[\|\delta_x v^1\|^2 - \alpha(\delta_x v^1, \delta_x v^0)] + (g^1, v^1) \\ &\leqslant -\tau^{-\alpha}\Big[\|\delta_x v^1\|^2 - (\|\delta_x v^1\|^2 + \frac{\alpha^2}{4}\|\delta_x v^0\|^2)\Big] + \frac{1}{4\tau}\|v^1\|^2 + \tau\|g^1\|^2 \\ &= \frac{\alpha^2}{4}\tau^{-\alpha}\|\delta_x v^0\|^2 + \frac{1}{4\tau}\|v^1\|^2 + \tau\|g^1\|^2.\end{aligned}$$

Multiplying both sides of the above inequality by 4τ, we have

$$\|v^1\|^2 \leqslant 2\|v^0\|^2 + \alpha^2\tau^{1-\alpha}\|\delta_x v^0\|^2 + 4\|\tau g^1\|^2. \tag{3.8}$$

Taking the inner product of Eq. (3.3) with v^k, using the summation formula by parts, we get

$$(D_{\hat{t}} v^k, v^k) = -\tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}(\delta_x v^{k-l}, \delta_x v^k) + (g^k, v^k), \quad 2 \leqslant k \leqslant N.$$

According to Lemma 4, it follows

$$\frac{1}{4\tau}(E^k - E^{k-1}) \leqslant -\tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}(\delta_x v^{k-l}, \delta_x v^k) + (g^k, v^k), \quad 2 \leqslant k \leqslant N.$$

Summing up for k from 2 to n gives

$$\frac{1}{4\tau}(E^n - E^1) \leqslant -\tau^{-\alpha}\sum_{k=2}^{n}\sum_{l=0}^{k} w_l^{(\alpha)}(\delta_x v^{k-l}, \delta_x v^k) + \sum_{k=2}^{n}(g^k, v^k), \quad 2 \leqslant n \leqslant N. \tag{3.9}$$

The adding of the inequality (3.9) and (3.7) leads to

$$\begin{aligned}&\frac{1}{4\tau}(E^n - E^1) + \frac{1}{2\tau}(\|v^1\|^2 - \|v^0\|^2) + \frac{\tau}{2}\|\delta_t v^{\frac{1}{2}}\|^2 \\ \leqslant& -\tau^{-\alpha}\sum_{k=0}^{n}\sum_{l=0}^{k} w_l^{(\alpha)}(\delta_x v^{k-l}, \delta_x v^k) + \sum_{k=2}^{n}(g^k, v^k) + (g^1, v^1) + \tau^{-\alpha}\|\delta_x v^0\|^2,\ 2 \leqslant n \leqslant N.\end{aligned}$$

It is indicated by Lemma 3 that

$$\tau^{-\alpha}\sum_{k=0}^{n}\sum_{l=0}^{k} w_l^{(\alpha)}(\delta_x v^{k-l}, \delta_x v^k) \geqslant 0,$$

and hence,

$$\begin{aligned}&\frac{1}{4\tau}(E^n - E^1) + \frac{1}{2\tau}(\|v^1\|^2 - \|v^0\|^2) + \frac{\tau}{2}\|\delta_t v^{\frac{1}{2}}\|^2 \\ \leqslant& \sum_{k=2}^{n}(g^k, v^k) + (g^1, v^1) + \tau^{-\alpha}\|\delta_x v^0\|^2,\ 2 \leqslant n \leqslant N.\end{aligned}$$

Multiplying 4τ on both sides of the above inequality and using $4(\tau g^1, v^1) \leqslant 4\|\tau g^1\|\|v^1\| \leqslant 4\|\tau g^1\|^2 + \|v^1\|^2$, we obtain

$$\begin{aligned}&E^n + 2\|v^1\|^2 + 2\|v^1 - v^0\|^2 \\ \leqslant& E^1 + 2\|v^0\|^2 + 4\tau\sum_{k=2}^{n}(g^k, v^k) + (4\|\tau g^1\|^2 + \|v^1\|^2) + 4\tau^{1-\alpha}\|\delta_x v^0\|^2 \\ =&(4-\alpha)\|v^1\|^2 + (1+\alpha)\|v^0\|^2 + 2\|v^1 - v^0\|^2 \\ &+ 4\tau\sum_{k=2}^{n}(g^k, v^k) + 4\|\tau g^1\|^2 + 4\tau^{1-\alpha}\|\delta_x v^0\|^2,\ 2 \leqslant n \leqslant N,\end{aligned}$$

hence,

$$E^n \leqslant (2-\alpha)\|v^1\|^2 + (1+\alpha)\|v^0\|^2 + 4\tau\sum_{k=2}^{n}(g^k, v^k) + 4\|\tau g^1\|^2 + 4\tau^{1-\alpha}\|\delta_x v^0\|^2,\ 2 \leqslant n \leqslant N. \tag{3.10}$$

We note that, according to Lemma 4, $\|v^n\|^2 \leqslant E^n,\ 1 \leqslant n \leqslant N$. So, the combination of (3.10) and (3.8) yields

$$\begin{aligned}\|v^n\|^2 \leqslant&(2-\alpha)\|v^1\|^2 + (1+\alpha)\|v^0\|^2 + 4\tau\sum_{k=2}^{n}(g^k, v^k) + 4\|\tau g^1\|^2 + 4\tau^{1-\alpha}\|\delta_x v^0\|^2\\ \leqslant&(5-\alpha)\|v^0\|^2 + [4+(2-\alpha)\alpha^2]\tau^{1-\alpha}\|\delta_x v^0\|^2 + 4(3-\alpha)\|\tau g^1\|^2 + 4\tau\sum_{k=2}^{n}(g^k, v^k)\\ \leqslant&(5-\alpha)\|v^0\|^2 + [4+(2-\alpha)\alpha^2]\tau^{1-\alpha}\|\delta_x v^0\|^2 + 4(3-\alpha)\|\tau g^1\|^2\\ &+ 4\tau\sum_{k=2}^{n}\|g^k\|^2 + \tau\sum_{k=2}^{n}\|v^k\|^2, \quad 2 \leqslant n \leqslant N. \end{aligned} \tag{3.11}$$

In addition, the inequality (3.8) indicates that (3.11) is also valid for $n = 1$. Hence, by the discrete Gronwall inequality [37], when $\tau \leqslant 1/2$, we have

$$\begin{aligned}\|v^n\|^2 \leqslant&\Big[(5-\alpha)\|v^0\|^2 + \Big(4+(2-\alpha)\alpha^2\Big)\tau^{1-\alpha}\|\delta_x v^0\|^2 + 4(3-\alpha)\|\tau g^1\|^2\\ &+ 4\tau\sum_{k=2}^{n}\|g^k\|^2\Big]\exp(2n\tau), \quad 1 \leqslant n \leqslant N.\end{aligned}$$

The proof ends. □

From Theorem 1, the following stability statement can be drawn.

Theorem 2. *The solution of finite difference scheme* (2.12)–(2.15) *is unconditionally stable with respect to the initial values and source terms.*

Next, the convergence of finite difference scheme (2.12)–(2.15) is analyzed. Denote

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N.$$

Theorem 3. *Suppose* $u(x,t) \in C^{(4,3)}([0,L]\times[0,T])$, $\partial^k u(x,0)/\partial t^k = 0$ *for* $k = 0,1,2$, *and* $\{U_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$, $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ *are solutions of the problem* (2.1)–(2.3), *and the finite difference scheme* (2.12)–(2.15), *respectively. Then it holds*

$$\|e^k\| \leqslant c_3(\tau^2 + h^2), \quad 0 \leqslant k \leqslant N, \tag{3.12}$$

where

$$c_3 = 2\exp(T)\sqrt{L\Big[(3-\alpha)c_2^2 + Tc_1^2\Big]}. \tag{3.13}$$

Proof. Subtracting (2.12)–(2.13) from the Eq. (2.8) together with (2.10) and noticing the exact initial-boundary value conditions, the error equations are resulted as

$$D_{\hat{t}}e_i^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 e_i^{k-l} + R_i^k,\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N, \tag{3.14}$$

$$\delta_t e_i^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\delta_x^2 e_i^{1-l} + R_i^1,\ 1 \leqslant i \leqslant M-1, \tag{3.15}$$

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant M, \tag{3.16}$$

$$e_0^k = e_M^k = 0,\ 1 \leqslant k \leqslant N. \tag{3.17}$$

When $\tau \leqslant 1/2$, the application of Theorem 1 into (3.14)–(3.17) gives

$$\|e^n\|^2 \leqslant \Big[4(3-\alpha)\|\tau R^1\|^2 + 4\tau\sum_{k=2}^{n}\|R^k\|^2\Big]\exp(2T), \quad 1 \leqslant n \leqslant N.$$

Using (2.9) and (2.11), further it follows

$$\|e^n\|^2 \leqslant 4L\exp(2T)\Big[(3-\alpha)c_2^2 + Tc_1^2\Big](\tau^2+h^2)^2, \quad 1 \leqslant n \leqslant N.$$

This proves the conclusion after taking the square root on both sides of the above inequality with c_3 given in (3.13). □

Remark: Readers may question that the errors on the first time level will be remembered through the memory effect of the fractional derivative, however, we note that the global second-order accuracy of the difference scheme (2.12)–(2.15) in time can be obtained in spite of only the first-order local truncation in time on the first time level $t = t_1$ because a small factor τ is multiplied with R^1 during the error analysis. Of course, if one would like to avoid the little losing of numerical accuracy at the beginning, $(2-\alpha)(u_i^1-u_i^0)/\tau-(1-\alpha)u_t(t_0)$ can be taken into approximating the values of $u_t(x_i, t_{1-\alpha/2})$. Whereas, it is unnecessary since the global numerical accuracy would not be influenced by the first-order truncation error on the first time level.

4. Improvement in the spatial approximation

In this section, the spatial compact approximation is brought here to improve the numerical accuracy of difference scheme in space. To aim this, for any mesh function $u, v \in \mathcal{V}_h$, denote

$$\mathcal{H}u_i = \begin{cases} (I + \frac{h^2}{12}\delta_x^2)u_i, & 1 \leqslant i \leqslant M-1, \\ u_i, & i = 0 \text{ or } i = M, \end{cases}$$

and introduce the discrete inner products and the corresponding norms as follows:

$$\langle u, v\rangle = (u, v) - \frac{h^2}{12}(\delta_x u, \delta_x v), \quad \|u\|_{\mathcal{H}} = \sqrt{\langle u, u\rangle}.$$

The following lemma will be used.

Lemma 5. *[18] Let function $g(x) \in C^6[0,L]$ and $\xi(\lambda) = (1-\lambda)^3[5-3(1-\lambda)^2]$. Then*

$$\mathcal{H}g''(x_i) = \delta_x^2 g(x_i) + \frac{h^4}{360}\int_0^1 [g^{(6)}(x_i - \lambda h) + g^{(6)}(x_i + \lambda h)]\xi(\lambda)\,\mathrm{d}\lambda, \quad 1 \leqslant i \leqslant M-1.$$

Implementing the compact operator $\mathcal{H}$ on both sides of Eq. (2.5), we get

$$\mathcal{H}u_t(x_i, t_{k-\frac{\alpha}{2}}) = {}_0^{RL}\mathcal{D}_t^{\alpha}\mathcal{H}u_{xx}(x_i, t_{k-\frac{\alpha}{2}}) + \mathcal{H}f(x_i, t_{k-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant N. \quad (4.1)$$

Assume $u(x,t) \in C^{(6,3)}([0,L]\times[0,T])$, $\partial^k u(x,0)/\partial t^k = 0$ for $k = 0,1,2$. By Lemma 2, Lemma 5, Eq. (2.4) and Taylor's expansion, we have

$$\mathcal{H}D_{\hat{t}}U_i^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 U_i^{k-l} + \mathcal{H}f(x_i, t_{k-\frac{\alpha}{2}}) + T_i^k,\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N, \quad (4.2)$$

$$\mathcal{H}\delta_t U_i^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\delta_x^2 U_i^{1-l} + \mathcal{H}f(x_i, t_{1-\frac{\alpha}{2}}) + T_i^1,\ 1 \leqslant i \leqslant M-1, \quad (4.3)$$

where there exists a positive constant c_4 such that

$$|T_i^1| \leqslant c_4(\tau + h^4), \quad |T_i^k| \leqslant c_4(\tau^2 + h^4), \quad 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N. \quad (4.4)$$

Neglecting small terms T_i^k in (4.2)–(4.3) and noting the initial-boundary value conditions (2.2)–(2.3), the fourth-order compact finite difference scheme in space for solving the problem (2.1)–(2.3) is established as follows:

$$\mathcal{H}D_{\hat{t}}u_i^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}\delta_x^2 u_i^{k-l} + \mathcal{H}f(x_i, t_{k-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1,\ 2 \leqslant k \leqslant N, \quad (4.5)$$

$$\mathcal{H}\delta_t u_i^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\delta_x^2 u_i^{1-l} + \mathcal{H}f(x_i, t_{1-\frac{\alpha}{2}}),\ 1 \leqslant i \leqslant M-1, \quad (4.6)$$

$$u_i^0 = 0, \quad 0 \leqslant i \leqslant M, \quad (4.7)$$

$$u_0^k = \phi_1(t_k), \quad u_M^k = \phi_2(t_k),\ 1 \leqslant k \leqslant N. \quad (4.8)$$

On each time level $t = t_k (1 \leqslant k \leqslant N)$, the difference scheme (4.5)–(4.8) is a tri-diagonal linear algebraic system with respect to the unknowns $\{u_i^k \mid 0 \leqslant i \leqslant M\}$ and the coefficient matrix is strictly diagonally dominant. Hence the solution is unique and it can be solved by Thomas algorithm.

For the sake of analysis on the compact difference scheme (4.5)–(4.8), several lemmas are listed.

Lemma 6. *[18] For $\forall\, v \in \mathcal{V}_h$, we have*

$$\frac{2}{3}\|v\|^2 \leqslant \langle v, v\rangle \leqslant \|v\|^2.$$

Lemma 7. *For any mesh function* $\{v_i^k \mid 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant N\}$ *with* $v_0^k = v_M^k = 0$ $(0 \leqslant k \leqslant N)$ *and* $0 < \alpha < 1$*, we have*

$$(\mathcal{H}D_{\hat{t}}v^k, v^k) \geqslant \frac{1}{4\tau}(E_{\mathcal{H}}^k - E_{\mathcal{H}}^{k-1}), \quad k \geqslant 2, \tag{4.9}$$

with

$$E_{\mathcal{H}}^k = (3-\alpha)\|v^k\|_{\mathcal{H}}^2 - (1-\alpha)\|v^{k-1}\|_{\mathcal{H}}^2 + 2\|v^k - v^{k-1}\|_{\mathcal{H}}^2, \quad k \geqslant 1.$$

In addition, it holds

$$E_{\mathcal{H}}^k \geqslant \|v^k\|_{\mathcal{H}}^2 \geqslant \frac{2}{3}\|v^k\|^2, \quad k \geqslant 1. \tag{4.10}$$

Proof. Noticing that $v_0^k = v_M^k = 0$, it holds for $2 \leqslant k \leqslant N$,

$$(\mathcal{H}D_{\hat{t}}v^k, v^k) = \Big((I + \frac{h^2}{12}\delta_x^2)D_{\hat{t}}v^k, v^k\Big) = (D_{\hat{t}}v^k, v^k) - \frac{h^2}{12}(\delta_x D_{\hat{t}}v^k, \delta_x v^k) = \langle D_{\hat{t}}v^k, v^k\rangle.$$

The direct application of Lemma 4 with the inner product $\langle\cdot,\cdot\rangle$ instead of $(\cdot,\cdot)$ will give the inequality (4.9). And the inequality (4.10) can be deduced by (3.2) and Lemma 6. □

Taking the inner product $(\cdot,\cdot)$ of (4.6) with u^1 and (4.5) with u^k $(2 \leqslant k \leqslant N)$, respectively, the similar process with Theorem 1 is carried out and using Lemma 6, Lemma 7, the unconditional stability of compact difference scheme (4.5)–(4.8) with respect to initial values and source terms can also be obtained.

The convergence of compact difference scheme (4.5)–(4.8) can be stated in the following theorem:

Theorem 4. *Suppose* $u(x,t) \in C^{(6,3)}([0,L]\times[0,T])$, $\partial^k u(x,0)/\partial t^k = 0$ *for* $k = 0,1,2$, *and* $\{U_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$, $\{u_i^k \mid 0 \leqslant i \leqslant M,\ 0 \leqslant k \leqslant N\}$ *are solutions of the problem* (2.1)–(2.3)*, and the compact finite difference scheme* (4.5)–(4.8)*, respectively. Then it holds*

$$\|e^k\| \leqslant c_5(\tau^2 + h^4), \quad 0 \leqslant k \leqslant N, \tag{4.11}$$

with c_5 *a positive constant dependent of* c_4, L, T *but independent of* h *and* τ*.*

Proof. Applying the similar argument as in the proof of Theorem 3, with the help of Lemma 6, Lemma 7 and (4.4), the conclusion can be easily derived. So the details are omitted here. □

5. Extension to the two-dimensional time-fractional sub-diffusion equations

In this section, the above ideas are extended to handle a class of two-dimensional time-fractional sub-diffusion equations as follows:

$$u_t(x,y,t) = {}_0^{RL}\mathcal{D}_t^{\alpha}\Delta u(x,y,t) + f(x,y,t), \quad (x,y) \in \Omega := (0,L_x)\times(0,L_y),\ t \in (0,T], \tag{5.1}$$

$$u(x,y,0) = 0, \quad (x,y) \in \overline{\Omega} = [0,L_x]\times[0,L_y], \tag{5.2}$$

$$u(x,y,t)|_{\partial\Omega} = \phi(x,y,t), \quad 0 < t \leqslant T, \tag{5.3}$$

where Δ is the Laplacian operator, L_x, L_y, T are three positive constants, $f(x,y,t)$ and $\phi(x,y,t)$ are given and smooth enough functions.

5.1. Derivation of the compact difference scheme

For numerical approximation of the solutions to the problem (5.1)–(5.3), the mesh partition is introduced. Denote $x_i = ih_x (0 \leqslant i \leqslant M_1)$, $y_j = jh_y (0 \leqslant j \leqslant M_2)$ with $h_x = L_x/M_1, h_y = L_y/M_2$ and M_1, M_2 are two positive integers. Let $\overline{\Omega}_h = \{(x_i, y_j) \,|\, 0 \leqslant i \leqslant M_1, 0 \leqslant j \leqslant M_2\}$, $\Omega_h = \overline{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \overline{\Omega}_h \cap \partial\Omega$. Then the domain $\overline{\Omega}$ is covered by $\overline{\Omega}_h$. For any mesh function $u_h = \{u_{i,j} \,|\, 0 \leqslant i \leqslant M_1, 0 \leqslant j \leq M_2\}$ defined on $\overline{\Omega}_h$, denote

$$\delta_x u_{i-\frac{1}{2},j} = \frac{1}{h_x}(u_{i,j} - u_{i-1,j}), \quad \delta_y u_{i,j-\frac{1}{2}} = \frac{1}{h_y}(u_{i,j} - u_{i,j-1}),$$

$$\delta_x^2 u_{i,j} = \frac{1}{h_x}\big(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j}\big), \quad \delta_y^2 u_{i,j} = \frac{1}{h_y}\big(\delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}}\big).$$

Introduce the following two compact operators for any function u_h defined on $\overline{\Omega}_h$:

$$\mathcal{A}_x u_{i,j} = \begin{cases} (I + \frac{h_x^2}{12}\delta_x^2) u_{i,j}, & 1 \leqslant i \leqslant M_1 - 1, \\ u_{i,j}, & i = 0 \text{ or } i = M_1, \end{cases} \quad 0 \leqslant j \leqslant M_2,$$

$$\mathcal{A}_y u_{i,j} = \begin{cases} (I + \frac{h_y^2}{12}\delta_y^2) u_{i,j}, & 1 \leqslant j \leqslant M_2 - 1, \\ u_{i,j}, & j = 0 \text{ or } j = M_2, \end{cases} \quad 0 \leqslant i \leqslant M_1.$$

Considering Eq. (5.1) at points $(x_i, y_j, t_{k-\alpha/2})$, we have

$$u_t(x_i, y_j, t_{k-\alpha/2}) = {}_0^{RL}\mathcal{D}_t^{\alpha} \Delta u(x_i, y_j, t_{k-\alpha/2}) + f(x_i, y_j, t_{k-\alpha/2}),$$
$$1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, 1 \leqslant k \leqslant N. \tag{5.4}$$

Performing the compact operators $\mathcal{A}_x\mathcal{A}_y$ on both sides of Eq. (5.4) and noticing the commutativity of the operator $\mathcal{A}_x$ and $\mathcal{A}_y$, we get

$$\mathcal{A}_x\mathcal{A}_y u_t(x_i, y_j, t_{k-\alpha/2}) = {}_0^{RL}\mathcal{D}_t^{\alpha}[\mathcal{A}_y\mathcal{A}_x u_{xx}(x_i, y_j, t_{k-\alpha/2}) + \mathcal{A}_x\mathcal{A}_y u_{yy}(x_i, y_j, t_{k-\alpha/2})]$$
$$+ \mathcal{A}_x\mathcal{A}_y f(x_i, y_j, t_{k-\alpha/2}), \ 1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, 1 \leqslant k \leqslant N. \tag{5.5}$$

Denote $U_{i,j}^k = u(x_i, y_j, t_k), f_{i,j}^{k-\alpha/2} = f(x_i, y_j, t_{k-\frac{\alpha}{2}})$. Assume $u(x, y, t) \in C^{(6,6,3)}(\Omega \times [0, T])$, $\partial^k u(x, y, 0)/\partial t^k = 0$ for $k = 0, 1, 2$. By Lemma 2, Lemma 5, (2.4) and Taylor's expansion, we obtain

$$\mathcal{A}_x\mathcal{A}_y D_{\hat{t}} U_{i,j}^k = \tau^{-\alpha} \sum_{l=0}^{k} w_l^{(\alpha)} (\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2) U_{i,j}^{k-l}$$
$$+ \mathcal{A}_x\mathcal{A}_y f_{i,j}^{k-\alpha/2} + S_{i,j}^k, \ 1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \ 2 \leqslant k \leqslant N, \tag{5.6}$$

$$\mathcal{A}_x\mathcal{A}_y \delta_t U_{i,j}^{\frac{1}{2}} = \tau^{-\alpha} \sum_{l=0}^{1} w_l^{(\alpha)} (\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2) U_{i,j}^{1-l}$$
$$+ \mathcal{A}_x\mathcal{A}_y f_{i,j}^{1-\alpha/2} + S_{i,j}^1, \ 1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \tag{5.7}$$

where there exists a positive constant c_6 such that

$$|S^1_{i,j}| \leqslant c_6(\tau + h_x^4 + h_y^4), \quad |S^k_{i,j}| \leqslant c_6(\tau^2 + h_x^4 + h_y^4), \quad 2 \leqslant k \leqslant N. \tag{5.8}$$

Omitting the small terms $S^k_{i,j}$ in Eqs. (5.6)–(5.7) and noticing the exact initial-boundary value conditions (5.2)–(5.3), the compact difference scheme for solving the problem (5.1)–(5.3) is derived as

$$\mathcal{A}_x\mathcal{A}_y D_{\hat{t}} u^k_{i,j} = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)u^{k-l}_{i,j} + \mathcal{A}_x\mathcal{A}_y f^{k-\alpha/2}_{i,j},$$
$$1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \ 2 \leqslant k \leqslant N, \tag{5.9}$$

$$\mathcal{A}_x\mathcal{A}_y \delta_t u^{\frac{1}{2}}_{i,j} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)u^{1-l}_{i,j} + \mathcal{A}_x\mathcal{A}_y f^{1-\alpha/2}_{i,j},$$
$$1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \tag{5.10}$$

$$u^0_{i,j} = 0, \quad 0 \leqslant i \leqslant M_1, \ 0 \leqslant j \leqslant M_2, \tag{5.11}$$

$$u^k_{i,j} = \phi(x_i, y_j, t_k), \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leqslant k \leqslant N. \tag{5.12}$$

5.2. Analysis on the compact difference scheme (5.9)–(5.12)

Define $\mathcal{W}_h = \{w \mid w = \{w_{i,j}\}$ is the mesh function defined on $\overline{\Omega}_h$ and $w|_{\partial\Omega_h} = 0\}$. For any mesh functions u, v defined on $\mathcal{W}_h$, define the inner products as

$$(u, v) = h_x h_y \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} u_{i,j}v_{i,j}, \quad (\delta_x\delta_y u, \delta_x\delta_y v) = h_x h_y \sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\left(\delta_x\delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}\right)\left(\delta_x\delta_y v_{i-\frac{1}{2},j-\frac{1}{2}}\right),$$

$$(\delta_x u, \delta_x v) = h_x h_y \sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1}\left(\delta_x u_{i-\frac{1}{2},j}\right)\left(\delta_x v_{i-\frac{1}{2},j}\right), \ (\delta_y u, \delta_y v) = h_x h_y \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2}\left(\delta_y u_{i,j-\frac{1}{2}}\right)\left(\delta_y v_{i,j-\frac{1}{2}}\right),$$

$$\lfloor u, v \rfloor = (\mathcal{A}_x\mathcal{A}_y u, v),$$

and the corresponding norms are denoted as

$$\|u\| = \sqrt{(u, u)}, \quad \|u\|_{\mathcal{A}} = \sqrt{\lfloor u, u \rfloor},$$

and the other norms are defined similarly. It is obvious that

$$\lfloor u, v \rfloor = (u, v) - \frac{h_y^2}{12}(\delta_y u, \delta_y v) - \frac{h_x^2}{12}(\delta_x u, \delta_x v) + \frac{h_x^2 h_y^2}{144}(\delta_x\delta_y u, \delta_x\delta_y v).$$

For the prior estimate of the compact difference scheme (5.9)–(5.12), a lemma is listed as follows.

Lemma 8. *[38] For any mesh function* v *defined on* $\mathcal{W}_h$*, it holds*

$$\frac{1}{3}\|v\|^2 \leqslant \|v\|_{\mathcal{A}}^2 \leqslant \|v\|^2.$$

In addition, it is easy to find that the compact operators $\mathcal{A}_x$ and $\mathcal{A}_y$ are both symmetric and positive definite, so their square root operators, which are denoted as Q_x and Q_y, respectively, (namely, $\mathcal{A}_x = Q_x^2, \mathcal{A}_y = Q_y^2$), are also both symmetric and positive definite.

Next, a prior estimate theorem is proved.

Theorem 5. *Suppose* $\{u_{i,j}^k \mid 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N\}$ *is the solution of the difference scheme*

$$\mathcal{A}_x\mathcal{A}_y D_{\hat{t}} u_{i,j}^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)u_{i,j}^{k-l} + g_{i,j}^k,$$
$$1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1,\ 2 \leqslant k \leqslant N, \tag{5.13}$$

$$\mathcal{A}_x\mathcal{A}_y\delta_t u_{i,j}^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)u_{i,j}^{1-l} + g_{i,j}^1, \quad 1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \tag{5.14}$$

$$u_{i,j}^0 = \psi(x_i, y_j), \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.15}$$

$$u_{i,j}^k = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leqslant k \leqslant N. \tag{5.16}$$

Then,

$$\|u^n\|^2 \leqslant \Big\{(39 - 15\alpha)\|u^0\|^2 + [12 + 9\alpha^2(2-\alpha)]\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2)$$
$$+ 12(19 - 9\alpha)\|\tau g^1\|^2 + 36\tau\sum_{k=2}^{n}\|g^k\|^2\Big\}\exp(2n\tau), \quad 1 \leqslant n \leqslant N. \tag{5.17}$$

Proof. Taking the inner product $(\cdot,\cdot)$ of Eq. (5.14) with u^1, noticing (5.16) and using the summation formula by parts, we get

$$\lfloor \delta_t u^{\frac{1}{2}}, u^1 \rfloor = \frac{1}{2\tau}(\|u^1\|_{\mathcal{A}}^2 - \|u^0\|_{\mathcal{A}}^2) + \frac{1}{2\tau}\|u^1 - u^0\|_{\mathcal{A}}^2$$
$$= -\tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\big[(\mathcal{A}_y\delta_x u^{1-l}, \delta_x u^1) + (\mathcal{A}_x\delta_y u^{1-l}, \delta_y u^1)\big] + (g^1, u^1)$$
$$= -\tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}\big[(Q_y\delta_x u^{1-l}, Q_y\delta_x u^1) + (Q_x\delta_y u^{1-l}, Q_x\delta_y u^1)\big] + (g^1, u^1). \tag{5.18}$$

Due to that

$$
\begin{aligned}
&-\sum_{l=0}^{1} w_l^{(\alpha)}\big[(Q_y\delta_x u^{1-l}, Q_y\delta_x u^1) + (Q_x\delta_y u^{1-l}, Q_x\delta_y u^1)\big] \\
&= -\|Q_y\delta_x u^1\|^2 - \|Q_x\delta_y u^1\|^2 + \alpha\big[(Q_y\delta_x u^0, Q_y\delta_x u^1) + (Q_x\delta_y u^0, Q_x\delta_y u^1)\big] \\
&\leqslant \frac{\alpha^2}{4}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2),
\end{aligned}
$$

and

$$
(g^1, u^1) \leqslant \frac{1}{12\tau}\|u^1\|^2 + 3\tau\|g^1\|^2,
$$

by Lemma 8, it follows

$$
\frac{1}{6\tau}\|u^1\|^2 \leqslant \frac{1}{2\tau}\|u^0\|^2 + \frac{1}{12\tau}\|u^1\|^2 + 3\tau\|g^1\|^2 + \frac{\alpha^2}{4}\tau^{-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2),
$$

namely,

$$
\|u^1\|^2 \leqslant 6\|u^0\|^2 + 36\|\tau g^1\|^2 + 3\alpha^2\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2). \tag{5.19}
$$

Taking the inner product $(\cdot,\cdot)$ of Eq. (5.13) with u^k, it yields

$$
\lfloor D_{\hat{t}}u^k, u^k\rfloor = -\tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}[(Q_y\delta_x u^{k-l}, Q_y\delta_x u^k) + (Q_x\delta_y u^{k-l}, Q_x\delta_y u^k)] + (g^k, u^k),\ 2 \leqslant k \leqslant N. \tag{5.20}
$$

It follows from the similar process of Lemma 7 that

$$
\lfloor D_{\hat{t}}u^k, u^k\rfloor \geqslant \frac{1}{4\tau}(F_{\mathcal{A}}^k - F_{\mathcal{A}}^{k-1}), \quad 2 \leqslant k \leqslant N,
$$

with

$$
F_{\mathcal{A}}^k = (3-\alpha)\|u^k\|_{\mathcal{A}}^2 - (1-\alpha)\|u^{k-1}\|_{\mathcal{A}}^2 + 2\|u^k - u^{k-1}\|_{\mathcal{A}}^2, \quad k \geqslant 1,
$$

and

$$
F_{\mathcal{A}}^k \geqslant \|u^k\|_{\mathcal{A}}^2 \geqslant \frac{1}{3}\|u^k\|^2,\ k \geqslant 1. \tag{5.21}
$$

Summing up for k from 2 to n in Eq. (5.20), it produces

$$
\begin{aligned}
\frac{1}{4\tau}(F_{\mathcal{A}}^n - F_{\mathcal{A}}^1) \leqslant &-\tau^{-\alpha}\sum_{k=2}^{n}\sum_{l=0}^{k} w_l^{(\alpha)}[(Q_y\delta_x u^{k-l}, Q_y\delta_x u^k) + (Q_x\delta_y u^{k-l}, Q_x\delta_y u^k)] \\
&+ \sum_{k=2}^{n}(g^k, u^k), \quad 2 \leqslant n \leqslant N.
\end{aligned} \tag{5.22}
$$

The adding of (5.18) with the inequality (5.22) gives

$$\begin{aligned}
&\frac{1}{4\tau}(F_{\mathcal{A}}^n - F_{\mathcal{A}}^1) + \frac{1}{2\tau}(\|u^1\|_{\mathcal{A}}^2 - \|u^0\|_{\mathcal{A}}^2) + \frac{1}{2\tau}\|u^1 - u^0\|_{\mathcal{A}}^2 \\
\leqslant & - \tau^{-\alpha} \sum_{k=0}^{n} \sum_{l=0}^{k} w_l^{(\alpha)} [(Q_y\delta_x u^{k-l}, Q_y\delta_x u^k) + (Q_x\delta_y u^{k-l}, Q_x\delta_y u^k)] \\
& + \sum_{k=2}^{n} (g^k, u^k) + (g^1, u^1) + \tau^{-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2), \ 2 \leqslant n \leqslant N.
\end{aligned}$$

By Lemma 3, we know

$$-\tau^{-\alpha} \sum_{k=0}^{n} \sum_{l=0}^{k} w_l^{(\alpha)} [(Q_y\delta_x u^{k-l}, Q_y\delta_x u^k) + (Q_x\delta_y u^{k-l}, Q_x\delta_y u^k)] \leqslant 0,$$

hence, it follows

$$\begin{aligned}
&\frac{1}{4\tau}(F_{\mathcal{A}}^n - F_{\mathcal{A}}^1) + \frac{1}{2\tau}(\|u^1\|_{\mathcal{A}}^2 - \|u^0\|_{\mathcal{A}}^2) + \frac{1}{2\tau}\|u^1 - u^0\|_{\mathcal{A}}^2 \\
\leqslant & \sum_{k=2}^{n} (g^k, u^k) + (g^1, u^1) + \tau^{-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2), \ 2 \leqslant n \leqslant N.
\end{aligned} \tag{5.23}$$

Multiplying 4τ on both sides of the inequality (5.23) and using $4\tau(g^1, u^1) \leqslant 4\|\tau g^1\|^2 + \|u^1\|^2$, it can be resulted that

$$F_{\mathcal{A}}^n \leqslant (2-\alpha)\|u^1\|_{\mathcal{A}}^2 + (1+\alpha)\|u^0\|_{\mathcal{A}}^2 + 4\tau \sum_{k=2}^{n} (g^k, u^k) + 4\|\tau g^1\|^2 + 4\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2).$$

According to Lemma 8, the inequality (5.21) and (5.19), further we have

$$\begin{aligned}
\|u^n\|^2 \leqslant & 3(2-\alpha)\|u^1\|^2 + 3(1+\alpha)\|u^0\|^2 + 12\tau \sum_{k=2}^{n} (g^k, u^k) \\
& + 12\|\tau g^1\|^2 + 12\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2) \\
\leqslant & (39 - 15\alpha)\|u^0\|^2 + 12(19 - 9\alpha)\|\tau g^1\|^2 + 12\tau \sum_{k=2}^{n} (g^k, u^k) \\
& + [12 + 9\alpha^2(2-\alpha)]\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2) \\
\leqslant & (39 - 15\alpha)\|u^0\|^2 + 12(19 - 9\alpha)\|\tau g^1\|^2 + 36\tau \sum_{k=2}^{n} \|g^k\|^2 + \tau \sum_{k=2}^{n} \|u^k\|^2 \\
& + [12 + 9\alpha^2(2-\alpha)]\tau^{1-\alpha}(\|Q_y\delta_x u^0\|^2 + \|Q_x\delta_y u^0\|^2), \quad 2 \leqslant n \leqslant N.
\end{aligned} \tag{5.24}$$

When $\tau \leqslant 1/2$, the application of the discrete Gronwall inequality into (5.24) immediately leads to the estimate (5.17). The conclusion is also valid for $n = 1$. The proof ends. $\square$

The prior estimate Theorem 5 reveals that the compact difference scheme (5.9)–(5.12) is unconditionally stable with respect to the initial values and source terms. Next, the convergence conclusion of this scheme is stated.

Denote

$$e_{i,j}^k = U_{i,j}^k - u_{i,j}^k, \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N.$$

Theorem 6. *Suppose $u(x,y,t) \in C^{(6,6,3)}(\Omega \times [0,T])$, $\partial^k u(x,y,0)/\partial t^k = 0$ for $k = 0,1,2$, and $\{U_{i,j}^k | 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N\}$, $\{u_{i,j}^k | 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2,\ 0 \leqslant k \leqslant N\}$ are solutions of the problem (5.1)–(5.3), and the compact finite difference scheme (5.9)–(5.12), respectively. Then it holds*

$$\|e^k\| \leqslant c_7(\tau^2 + h_x^4 + h_y^4), \quad 0 \leqslant k \leqslant N, \tag{5.25}$$

with c_7 a positive constant dependent of c_6, L_x, L_y, T but independent of h_x, h_y and τ.

Proof. Subtracting the Eqs. (5.9)–(5.10) from the Eqs. (5.6)–(5.7) and noticing the exact initial-boundary value conditions, the error equations are obtained as

$$\mathcal{A}_x\mathcal{A}_y D_{\hat{t}} e_{i,j}^k = \tau^{-\alpha}\sum_{l=0}^{k} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)e_{i,j}^{k-l} + S_{i,j}^k,$$
$$1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1,\ 2 \leqslant k \leqslant N, \tag{5.26}$$

$$\mathcal{A}_x\mathcal{A}_y \delta_t e_{i,j}^{\frac{1}{2}} = \tau^{-\alpha}\sum_{l=0}^{1} w_l^{(\alpha)}(\mathcal{A}_y\delta_x^2 + \mathcal{A}_x\delta_y^2)e_{i,j}^{1-l} + S_{i,j}^1,\ 1 \leqslant i \leqslant M_1 - 1, 1 \leqslant j \leqslant M_2 - 1, \tag{5.27}$$

$$e_{i,j}^0 = 0, \quad 0 \leqslant i \leqslant M_1,\ 0 \leqslant j \leqslant M_2, \tag{5.28}$$

$$e_{i,j}^k = 0, \quad (x_i, y_j) \in \partial\Omega_h, \quad 1 \leqslant k \leqslant N. \tag{5.29}$$

The application of Theorem 5 into the system (5.26)–(5.29) together with the inequality (5.8) will lead to the convergence conclusion (5.25). The proof ends. $\square$

6. Numerical examples

In this section, we shall test the computational efficiency of new developed difference schemes (2.12)–(2.15), (4.5)–(4.8) for the one-dimensional problem (2.1)–(2.3) and the compact difference scheme (5.9)–(5.12) for the two-dimensional problem (5.1)–(5.3).

6.1. The one-dimensional case

Denote

$$E^N(h,\tau) = \|U^N - u^N\|, \quad rate_\tau = \log_2 \frac{E^N(h,\tau)}{E_N(h,\tau/2)}, \quad rate_h = \frac{\log[E^N(h_1,\tau)/E^N(h_2,\tau)]}{\log(h_1/h_2)}.$$

Example 1. Take $L = 1$, $T = 1$, $f(x,t) = \sin(x)\Big[(3+\alpha)t^{2+\alpha} + \Gamma(4+\alpha)t^3/6\Big]$, $\phi_1(t) = 0$, $\phi_2(t) = t^{3+\alpha}\sin(1)$ in the problem (2.1)–(2.3).

The exact solution of Example 1 is given by $u(x,t) = t^{3+\alpha}\sin(x)$. Firstly, the numerical accuracies of both the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8) in time are tested. For different α $(0 < \alpha < 1)$, the numerical results are computed using the developed difference schemes with varying temporal stepsizes and sufficiently small spatial stepsizes. The second-order convergence of both the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8) in time can be observed from the data of Table 1.

Table 1: (Ex.1) Discrete L_2 norm errors and convergence orders of difference schemes in time.

		scheme (2.12)–(2.15)($M = 1000$)		*scheme* (4.5)–(4.8)($M = 100$)	
	τ	$E^N(h,\tau)$	$rate_\tau$	$E^N(h,\tau)$	$rate_\tau$
$\alpha = 0.05$	1/10	8.661134×10^{-4}	2.00	8.661106×10^{-4}	2.00
	1/20	2.172018×10^{-4}	2.00	2.171991×10^{-4}	2.00
	1/40	5.438462×10^{-5}	2.00	5.438188×10^{-5}	2.00
	1/80	1.360804×10^{-5}	2.00	1.360530×10^{-5}	2.00
	1/160	3.405248×10^{-6}	—	3.402519×10^{-6}	—
$\alpha = 0.5$	1/10	4.105762×10^{-4}	1.94	4.105732×10^{-4}	1.94
	1/20	1.068597×10^{-4}	1.97	1.068567×10^{-4}	1.97
	1/40	2.725799×10^{-5}	1.99	2.725499×10^{-5}	1.99
	1/80	6.885561×10^{-6}	1.99	6.882565×10^{-6}	1.99
	1/160	1.732349×10^{-6}	—	1.729352×10^{-6}	—
$\alpha = 0.95$	1/10	3.204606×10^{-4}	2.01	3.204638×10^{-4}	2.01
	1/20	7.963433×10^{-5}	2.00	7.963758×10^{-5}	2.00
	1/40	1.984454×10^{-5}	2.00	1.984781×10^{-5}	2.00
	1/80	4.950842×10^{-6}	2.00	4.954143×10^{-6}	2.00
	1/160	1.234187×10^{-6}	—	1.237551×10^{-6}	—

Secondly, the numerical accuracies of the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8) in space are verified by the example, respectively. With sufficiently small temporal stepsizes, the discrete L_2 norm of errors and numerical convergence orders of these schemes are illustrated in Table 2. The second-order convergence of difference scheme (2.12)–(2.15) in space and the fourth-order one for the scheme (4.5)–(4.8) can be seen from the table. Those are in good agreement with the theoretical results.

Table 2: (Ex.1) Discrete L_2 norm errors and convergence orders of difference schemes in space.

	h	*scheme* (2.12)–(2.15)($N = 1000$) $E^N(h,\tau)$	$rate_h$	*scheme* (4.5)–(4.8)($N = 100,000$) $E^N(h,\tau)$	$rate_h$
$\alpha = 0.25$	1/4	2.389627×10^{-4}	1.96	5.563999×10^{-7}	4.00
	1/8	6.163069×10^{-5}	1.98	3.481249×10^{-8}	4.00
	1/16	1.558938×10^{-5}	1.97	2.182438×10^{-9}	3.92
	1/32	3.987590×10^{-6}	—	1.437785×10^{-10}	—
$\alpha = 0.5$	1/4	2.524547×10^{-4}	1.96	5.853149×10^{-7}	4.00
	1/8	6.471947×10^{-5}	1.98	3.661277×10^{-8}	4.00
	1/16	1.635635×10^{-5}	1.98	2.291784×10^{-9}	3.96
	1/32	4.140753×10^{-6}	—	1.473553×10^{-10}	—
$\alpha = 0.75$	1/4	2.651721×10^{-4}	1.97	6.125700×10^{-7}	4.00
	1/8	6.761348×10^{-5}	1.99	3.830850×10^{-8}	4.00
	1/16	1.706267×10^{-5}	2.00	2.393163×10^{-9}	4.00
	1/32	4.268651×10^{-6}	—	1.495545×10^{-10}	—

Example 2. Take $L = 1$, $T = 1$, $f(x,t) = 2\sin(x)\Big[t + t^{2-\alpha}/\Gamma(3-\alpha)\Big]$, $\phi_1(t) = 0$, $\phi_2(t) = t^2\sin(1)$ in the problem (2.1)–(2.3).

The exact solution is given by $u(x,t) = t^2\sin(x)$. One can find that the condition $\partial^2 u(x,0)/\partial t^2 = 0$ in convergence Theorem 3 and Theorem 4 is not satisfied any longer. Now we would like to test the efficiency of the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8) for this case.

Firstly, with varying temporal stepsizes and the fixed sufficiently small spatial stepsizes, the example is computed using these two group of difference schemes. Table 3 listed the numerical results for different parameter α. Just as we hope, the second-order convergence of these two group of difference schemes in time can still be achieved. The results tell us that maybe the conditions of the above convergence theorems can be relaxed a little. Probably, the conditions of these convergence theorems are only sufficient but not necessary.

Next, fixing the sufficiently small temporal stepsizes, the example is computed under the varying spatial stepsizes using the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8), respectively. From Table 4, the spatial numerical convergence orders of these two group of difference schemes are two and fourth, respectively.

Finally, to further know about the roles of the conditions of $\partial^k u(x,0)/\partial t^k = 0$, $k = 0, 1, 2$, in convergence Theorem 3 and Theorem 4, we compute more examples by using the difference scheme (2.12)–(2.15) and the scheme (4.5)–(4.8), respectively. Table 5 and Table 6 report the numerical errors and convergence orders in time for the problems with the exact solutions $u(x,t) = \sin(x)t^{3/2}$ and $u(x,t) = \sin(x)t$ on the domain $(x,t) \in [0,1] \times [0,1]$, respectively. The second-order convergence of these two groups of difference schemes in time can not be achieved for the above two problems. From Table 3, Table 5 and Table 6, we can

Table 3: (Ex.2) Discrete L_2 norm errors and convergence orders of difference schemes in time.

	τ	*scheme* (2.12)–(2.15)($M = 1000$) $E^N(h,\tau)$	$rate_\tau$	*scheme* (4.5)–(4.8)($M = 100$) $E^N(h,\tau)$	$rate_\tau$
	1/10	3.330120×10^{-5}	1.93	3.329824×10^{-5}	1.93
	1/20	8.738860×10^{-6}	1.97	8.735909×10^{-6}	1.97
$\alpha = 0.05$	1/40	2.234821×10^{-6}	1.97	2.231866×10^{-6}	1.97
	1/80	5.715565×10^{-7}	1.96	5.686000×10^{-7}	1.98
	1/160	1.469684×10^{-7}	—	1.440174×10^{-7}	—
	1/10	1.184407×10^{-4}	2.01	1.184376×10^{-4}	2.01
	1/20	2.931647×10^{-5}	1.99	2.931336×10^{-5}	1.99
$\alpha = 0.5$	1/40	7.370752×10^{-6}	1.98	7.367644×10^{-6}	1.98
	1/80	1.869817×10^{-6}	1.97	1.866709×10^{-6}	1.98
	1/160	4.777724×10^{-7}	—	4.746646×10^{-7}	—
	1/10	1.398361×10^{-5}	2.10	1.398036×10^{-5}	2.10
	1/20	3.272140×10^{-6}	2.09	3.268869×10^{-6}	2.10
$\alpha = 0.95$	1/40	7.665694×10^{-7}	2.08	7.632733×10^{-7}	2.10
	1/80	1.812309×10^{-7}	2.02	1.778814×10^{-7}	2.10
	1/160	4.481091×10^{-8}	—	4.135993×10^{-8}	—

conclude that the conditions of $\partial^k u(x,0)/\partial t^k = 0$, $k = 0, 1, 2$, in convergence Theorem 3 and Theorem 4 are only sufficient but not necessary. Despite all this, some conditions are still indispensable.

6.2. The two-dimensional case

For simplicity, take $h_x = h_y = \widehat{h}$. Denote

$$\widehat{E}^N(\widehat{h},\tau) = \|U^N - u^N\|, \quad \widehat{rate}_\tau = \log_2 \frac{\widehat{E}^N(\widehat{h},\tau)}{\widehat{E}_N(\widehat{h},\tau/2)}, \quad \widehat{rate}_{\widehat{h}} = \log_2 \frac{\widehat{E}^N(\widehat{h},\tau)}{\widehat{E}_N(\widehat{h}/2,\tau)}.$$

Example 3. Take $L_x = L_y = \pi$, $T = 1$, $f(x,y,t) = 3t^2 \sin(x)\sin(y)\Big[1 + 4t^{1-\alpha}/\Gamma(4-\alpha)\Big]$, $\phi(x,y,t) = 0$ in the problem (5.1)–(5.3).

The analytic solution of the example is given by $u(x,y,t) = t^3 \sin(x)\sin(y)$. To verify the computational efficiency of the compact difference scheme (5.9)–(5.12), firstly, the example is numerically computed with the fixed and sufficiently small value $\widehat{h} = \pi/100$ for different anomalous diffusion exponent α. The computational results are shown in Table 7, from which, the global second-order convergence of the scheme (5.9)–(5.12) in time can be obtained.

Secondly, the spatial fourth-order convergence of the compact difference scheme (5.9)–(5.12) is tested. Taking the fixed $\tau = 1/10,000$, the example is computed using the scheme

Table 4: (Ex.2) Discrete L_2 norm errors and convergence orders of difference schemes in space.

	h	*scheme* (2.12)–(2.15)($N=1000$) $E^N(h,\tau)$	$rate_h$	*scheme* (4.5)–(4.8)($N=100,000$) $E^N(h,\tau)$	$rate_h$
$\alpha=0.1$	1/8	4.641567×10^{-5}	2.00	3.633648×10^{-8}	4.00
	1/12	2.065387×10^{-5}	2.00	7.176401×10^{-9}	4.00
	1/16	1.162460×10^{-5}	2.00	2.270961×10^{-9}	4.00
	1/20	7.443296×10^{-6}	2.00	9.303180×10^{-10}	4.00
	1/24	5.171445×10^{-6}	—	4.490515×10^{-10}	—
$\alpha=0.5$	1/8	4.854911×10^{-5}	2.00	3.798079×10^{-8}	4.00
	1/12	2.159890×10^{-5}	2.00	7.501857×10^{-9}	4.00
	1/16	1.215749×10^{-5}	2.00	2.374319×10^{-9}	3.99
	1/20	7.786087×10^{-6}	2.00	9.738994×10^{-10}	4.00
	1/24	5.411150×10^{-6}	—	4.699152×10^{-10}	—
$\alpha=0.95$	1/8	5.067421×10^{-5}	2.00	3.963620×10^{-8}	4.00
	1/12	2.253086×10^{-5}	2.00	7.827329×10^{-9}	4.02
	1/16	1.267537×10^{-5}	2.00	2.461372×10^{-9}	4.06
	1/20	8.112927×10^{-6}	2.00	9.953605×10^{-10}	4.22
	1/24	5.634377×10^{-6}	—	4.608665×10^{-10}	—

(5.9)–(5.12) and the numerical results are displayed in Table 8, which are in good agreement with the conclusion of Theorem 6.

7. Conclusion

In the present work, the superconvergence of the first-order standard Grünwald-Letnikov formula for approximating the Riemann-Liouville time-fractional derivative at some particular points is applied to develop the higher-order difference schemes for a class of time-fractional sub-diffusion equations. For one-dimensional case, two groups of difference schemes are investigated, one is second-order accurate both in time and space, and the other is fourth-order compact approximation in space. Strict stability and convergence analysis are given for these schemes. Then the extension to the two-dimensional problem is considered. Numerical examples illustrate the effectiveness and robustness of the methods. It is noted that the resultant schemes are easy to be implemented. The application of this novel ideas into the more general problems and the superconvergence of the other fractional numerical differentiation formulae will be our next consideration.

Acknowledgments

We would like to express our gratitude to the editor and reviewers for their valuable comments and suggestions.

Table 5: Discrete L_2 norm errors and convergence orders of difference schemes in time for the problem (2.1)–(2.3) with $u(x,t) = \sin(x)t^{3/2}$.

	τ	*scheme* (2.12)–(2.15)($M = 1000$) $E^N(h,\tau)$	$rate_\tau$	*scheme* (4.5)–(4.8)($M = 100$) $E^N(h,\tau)$	$rate_\tau$
$\alpha = 0.25$	1/10	$8.536513E-05$	1.32	$8.536199E-05$	1.32
	1/20	$3.413968E-05$	1.34	$3.413656E-05$	1.34
	1/40	$1.346803E-05$	1.37	$1.346492E-05$	1.37
	1/80	$5.198138E-06$	1.41	$5.195029E-06$	1.41
	1/160	$1.962103E-06$	—	$1.958994E-06$	—
$\alpha = 0.5$	1/10	$1.577985E-04$	1.50	$1.577953E-04$	1.50
	1/20	$5.572529E-05$	1.48	$5.572212E-05$	1.48
	1/40	$2.003031E-05$	1.46	$2.002715E-05$	1.46
	1/80	$7.259573E-06$	1.46	$7.256417E-06$	1.46
	1/160	$2.639529E-06$	—	$2.636374E-06$	—
$\alpha = 0.75$	1/10	$1.798318E-04$	1.53	$1.798286E-04$	1.53
	1/20	$6.227459E-05$	1.52	$6.227138E-05$	1.52
	1/40	$2.170250E-05$	1.51	$2.169930E-05$	1.52
	1/80	$7.594121E-06$	1.51	$7.590918E-06$	1.51
	1/160	$2.666134E-06$	—	$2.662929E-06$	—
$\alpha = 0.9$	1/10	$1.873737E-04$	1.50	$1.873704E-04$	1.50
	1/20	$6.613070E-05$	1.51	$6.612748E-05$	1.51
	1/40	$2.324558E-05$	1.51	$2.324238E-05$	1.51
	1/80	$8.148539E-06$	1.51	$8.145368E-06$	1.52
	1/160	$2.852160E-06$	—	$2.849049E-06$	—

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Table 6: Discrete L_2 norm errors and convergence orders of difference schemes in time for the problem (2.1)–(2.3) with $u(x,t) = \sin(x)t$.

	τ	*scheme* (2.12)–(2.15)$(M = 1000)$		*scheme* (4.5)–(4.8)$(M = 100)$	
		$E^N(h,\tau)$	$rate_\tau$	$E^N(h,\tau)$	$rate_\tau$
$\alpha = 0.25$	1/10	$1.585879E-05$	1.71	$1.586200E-05$	1.71
	1/20	$4.837879E-06$	1.72	$4.841087E-06$	1.72
	1/40	$1.465878E-06$	1.73	$1.469086E-06$	1.73
	1/80	$4.406771E-07$	1.76	$4.438849E-07$	1.73
	1/160	$1.304076E-07$	—	$1.336156E-07$	—
$\alpha = 0.5$	1/10	$4.015579E-05$	1.47	$4.015900E-05$	1.47
	1/20	$1.453507E-05$	1.48	$1.453828E-05$	1.48
	1/40	$5.225870E-06$	1.48	$5.229082E-06$	1.48
	1/80	$1.871307E-06$	1.49	$1.874519E-06$	1.48
	1/160	$6.673944E-07$	—	$6.706071E-07$	—
$\alpha = 0.75$	1/10	$5.673761E-05$	1.23	$5.674084E-05$	1.23
	1/20	$2.426822E-05$	1.24	$2.427145E-05$	1.24
	1/40	$1.030253E-05$	1.24	$1.030576E-05$	1.24
	1/80	$4.356861E-06$	1.24	$4.360088E-06$	1.24
	1/160	$1.838200E-06$	—	$1.841429E-06$	—
$\alpha = 0.9$	1/10	$4.023310E-05$	1.08	$4.023633E-05$	1.08
	1/20	$1.904750E-05$	1.09	$1.905072E-05$	1.09
	1/40	$8.951291E-06$	1.09	$8.954496E-06$	1.09
	1/80	$4.190619E-06$	1.10	$4.193784E-06$	1.10
	1/160	$1.957524E-06$	—	$1.960616E-06$	—

Table 7: (Ex.3) Discrete L_2 norm errors and convergence orders of the difference scheme (5.9)–(5.12) in time with $\widehat{h} = \pi/100$.

τ	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	$\widehat{E}^N(\widehat{h},\tau)$	$\widehat{rate}_\tau$	$\widehat{E}^N(\widehat{h},\tau)$	$\widehat{rate}_\tau$	$\widehat{E}^N(\widehat{h},\tau)$	$\widehat{rate}_\tau$
1/5	2.779127×10^{-2}	1.99	7.118857×10^{-3}	1.98	7.175418×10^{-3}	2.00
1/10	6.993052×10^{-3}	2.00	1.804082×10^{-3}	1.99	1.795015×10^{-3}	2.00
1/20	1.750978×10^{-3}	2.00	4.536174×10^{-4}	2.00	4.488967×10^{-4}	2.00
1/40	4.379483×10^{-4}	2.00	1.136747×10^{-4}	2.00	1.122393×10^{-4}	2.00
1/80	1.095032×10^{-4}	—	2.844719×10^{-5}	—	2.805981×10^{-5}	—

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Table 8: (Ex.3) Discrete L_2 norm errors and convergence orders of the difference scheme (5.9)–(5.12) in space with $\tau = 1/10,000$.

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
$\widehat{h}$	$\widehat{E}^N(\widehat{h}, \tau)$	$\widehat{rate}_{\widehat{h}}$	$\widehat{E}^N(\widehat{h}, \tau)$	$\widehat{rate}_{\widehat{h}}$	$\widehat{E}^N(\widehat{h}, \tau)$	$\widehat{rate}_{\widehat{h}}$
$\pi/4$	6.246267×10^{-4}	4.03	8.384381×10^{-4}	4.03	1.036662×10^{-3}	4.03
$\pi/8$	3.836520×10^{-5}	4.00	5.147831×10^{-5}	4.01	6.363237×10^{-5}	4.01
$\pi/16$	2.393530×10^{-6}	3.94	3.204474×10^{-6}	3.99	3.957211×10^{-6}	4.01
$\pi/32$	1.559952×10^{-7}	—	2.017578×10^{-7}	—	2.453512×10^{-7}	—

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