Some high-order difference schemes for the distributed-order differential equations

Guang-hua Gao*
College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210046, P.R. China

Hai-wei Sun†
Department of Mathematics, University of Macau, Macao, P.R. China

Zhi-zhong Sun‡
Department of Mathematics, Southeast University, Nanjing 210096, P.R. China

Abstract

Two difference schemes are derived for both one-dimensional and two-dimensional distributed-order differential equations. It is proved that the schemes are unconditionally stable and convergent in a mean L_∞ norm with the convergence orders of $O(\tau^2 + h^2 + \Delta\alpha^2)$ and $O(\tau^2 + h^4 + \Delta\alpha^4)$, respectively, where τ, h and $\Delta\alpha$ are the step sizes in time, space and distributed-order variables. Several numerical examples are given to confirm the theoretical results.

Keywords: distributed-order differential equations, high-order approximation, fractional derivative, difference scheme, stability, convergence

1 Introduction

In recent years, the research on fractional calculus has gained a growth interest due to its powerful potential to depict many processes in physics, engineering, finance, material science, control system, signal processing and so on. The related fractional differential equation has been studied from different viewpoints. For most of them, analytical solutions are not available or too complicated to compute, so some effective numerical methods are resorted to and there have been quite abundant literatures on this subject, in which the finite difference method is one popular way.

For the single-order time-fractional diffusion equation, the finite difference methods have been well studied in recent few years. We now here only mention a few part of related publications, such as the work by Yuste and Acedo [1], Langlands and Henry [2], Chen et al. [3], [4], Cao and Xu [5], Li and Ding [6], Wu and Sun [7], Wang and Vong [8]. For more results, readers can refer to the review article [9] and references therein.

*E-mail address: gaoguanghua1107@163.com
†E-mail address: hsun@unmc.mo
‡Corresponding author; E-mail address: zzsun@seu.edu.cn

It is pointed out recently that the time-fractional anomalous diffusion equation with a constant-order temporal derivative cannot describe the processes lacking temporal scaling, so the time-fractional diffusion equation of distributed order was introduced to describe processes getting more anomalous in course of time, i.e. the accelerating or retarded sub-diffusion [10]. The distributed-order differential equation can also be regraded as a natural generalization of the single order and multi-term fractional differential equation [11]. The idea of distributed-order differential equation was first proposed probably by Caputo in [12], [13], which was also stated by Podlubny et al. in [14]. Jiao, Chen and Podlubny [15] presented a concise and insightful view to understand the usefulness of distributed-order concept in control and signal processing. Luchko [16] investigated some uniqueness and existence results of boundary value problems for the generalized time-fractional diffusion equation of distributed order by an appropriate maximum principle. Gorenflo et al. [17] obtained a representation of the fundamental solution to the Cauchy problem by employing the technique of the Fourier and Laplace transforms and gave the interpretation of the fundamental solution as a probability density function. For more general distributed-order differential equations, analytical solution is not easy to obtain. Therefore, the consideration on the numerical method to solve the distributed-order differential equation should be involved.

Generally speaking, when numerically solving the distributed-order differential equations, the approximation to the integral of distributed-order variable is taken into account one step ahead, using the classical numerical quadrature formulae. Then the original distributed-order problem is approximated by a multi-term time-fractional differential problem. What follows is how to efficiently solve the approximated multi-term fractional differential problem. So far, to our knowledge, only a few works have been concerned with this issue. Podlubny et al. [14] proposed a matrix approach method which can be used to solve the distributed-order ordinary differential equation. Diethelm and Ford [11] provided some numerical analysis for distributed-order differential equation, where the equivalent system of single-term differential equations was used to treat the multi-term time-fractional differential problem after the numerical approximation by the quadrature formula for the distributed-order integral. Then a fractional linear multi-step method or an Adams-type predictor-corrector scheme was brought to solve the equivalent system. The similar techniques to deal with the multi-term fractional differential equation are discussed in recent work [18]. Recently, Ye et al. [19] constructed an implicit difference scheme for the time distributed-order and Riesz space fractional diffusions on bounded domains. The unconditional stability and convergence was proved by mathematical induction method. They continued their study for distributed-order time-fractional diffusion-wave equation on bounded domains in [20]. The equation is approximated by a multi-term fractional diffusion-wave equation, which is then solved by a compact difference scheme. Ford, Morgado and Rebelo [21] discussed a numerical method for the distributed-order time-fractional diffusion equation. Morgado and Rebelo [22] presented an implicit scheme for the numerical approximation of the distributed order time-fractional reaction-diffusion equation with a nonlinear source term. The midpoint rule was used to approximate the distribute integral in [19], [21] and [22], and the same discrete L1 formula was used to approximate the involved Caputo fractional derivatives, thus the numerical accuracies of the resultant schemes in time variable were all about first order. Katsikadelis [23] presented a numerical method for distributed-order differential equations, where the trapezoidal rule was used to discretize the integral with respect to the distributed-order variable and the analog equation method was employed to solve the approximated multi-term fractional differential equation. The stability and convergence were only shown by numerical examples without any strict proof. To our knowledge, the high-order schemes for numerically solving the distributed-order differential equations have not been seen.

In addition, the bottleneck of numerical methods for solving the time-fractional differential problem lies in the global storage and computation of unknowns at all previous time layers when the problem is approximated at the considered time layer, which is caused by the nonlocal property of fractional operators. So the high order numerical methods have been pursued for alleviating this drawback. Spatially compact techniques have been widely used to enhance the space numerical accuracy when solving the time-fractional differential problem. Major results along this routine cover the work by Cui [25], [26], Chen et al. [27], Hu and Zhang [28], [29], Sun et al. [30], [31], [32], [33] and so on.

Our work here will apply the weighted Grünwald formula, which was proposed in [24], to approximate the involved time-fractional derivatives. The numerical accuracy in time can achieve a global second order independent of orders of fractional derivatives. On the other hand, we will also make our attempt to attack the numerical solutions with high-order accuracy in space for solving the distributed-order differential equation.

Consider the following described distributed-order problem

$$\mathcal{D}_t^w u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + F(\mathbf{x},t), \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{1.1}$$

$$u(\mathbf{x},0) = 0, \quad x \in \overline{\Omega}, \tag{1.2}$$

$$u(\mathbf{x},t)|_{\mathbf{x}\in\partial\Omega} = \psi(\mathbf{x},t), \quad 0 < t \leqslant T, \tag{1.3}$$

where $\partial\Omega$ is the boundary of Ω, $F(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ are given smooth functions,

$$\psi(\mathbf{x},0) = 0, \text{ if } \mathbf{x} \in \partial\Omega, \qquad \mathcal{D}_t^w u(x,t) = \int_0^1 w(\alpha)\, {}_0^C D_t^\alpha u(x,t) d\alpha,$$

$$w(\alpha) \geqslant 0, \qquad \int_0^1 w(\alpha) d\alpha = c_0 > 0,$$

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u}{\partial \xi}(x,\xi) d\xi, \quad 0 \leqslant \alpha < 1, \; {}_0^C D_t^\alpha u(x,t) = u_t(x,t) \text{ for } \alpha = 1.$$

It is noted that the zero initial case is only considered here. If it is not so, then an auxiliary unknown function can be introduced to return it to the zero intial problem.

To illustrate the idea of approximation, a second-order accurate difference scheme in all variables is derived firstly. Then the spatially compact scheme will also be established for the distributed-order diffusion equation. To match the fourth-order high accuracy, the Simpson numerical quadrature formula is used to discretize the integral with respect to the distributed-order variable. Hence, a new fourth-order difference scheme in both space and distributed-order variables is developed and the unconditional stability as well as convergence are proved.

The plan of this work is as follows. In section 2, some preliminary numerical quadrature formulae and useful lemmas are prepared. In section 3, the distributed-order problem (1.1)–(1.3) in one-dimensional case is discussed. Two difference schemes are derived along with the stability and convergence analysis. In section 4, we turn to treating the two-dimensional distributed-order problem (1.1)–(1.3). The space second-order accurate and fourth-order compact schemes are established, respectively. The strict stability and convergence analysis in a mean maximum norm are proved using the discrete energy method. Section 5 focuses on the displaying of several numerical examples to show the effectiveness of the developed methods. A brief conclusion ends this work finally.

2 Preliminary

Divide the interval $[0,1]$ into $2J$-subintervals with $\triangle\alpha = \frac{1}{2J}$ and $\alpha_l = l\triangle\alpha, l = 0,1,2,\dots,2J$. Divide the interval $[0,T]$ into N-subintervals with $\tau = \frac{T}{N}$ and $t_k = k\tau, k = 0,1,2,\dots,N$.

For the numerical approximation, several formulae and lemmas are prepared below.

The composite trapezoid formula

Lemma 2.1 *Let $s(\alpha) \in C^{(2)}[0,1]$. Then we have*

$$\int_0^1 s(\alpha)d\alpha = \triangle\alpha \sum_{l=0}^{2J} c_l s(\alpha_l) - \frac{\triangle\alpha^2}{12} s^{(2)}(\xi), \quad \xi \in (0,1),$$

where

$$c_l = \begin{cases} \frac{1}{2}, & l = 0,\ 2J, \\ 1, & l = 1,2,3,\dots,2J-2,2J-1. \end{cases}$$

The composite Simpson formula

Lemma 2.2 *Let $s(\alpha) \in C^{(4)}[0,1]$. Then we have*

$$\int_0^1 s(\alpha)d\alpha = \triangle\alpha \sum_{l=0}^{2J} d_l s(\alpha_l) - \frac{\triangle\alpha^4}{180} s^{(4)}(\eta), \quad \eta \in (0,1),$$

where

$$d_l = \begin{cases} \frac{1}{3}, & l = 0,\ 2J, \\ \frac{2}{3}, & l = 2,\ 4,\ \dots,\ 2J-4,\ 2J-2, \\ \frac{4}{3}, & l = 1,\ 3,\ \dots,\ 2J-3,\ 2J-1. \end{cases}$$

Grünwald formula

Let $f \in L^1(\mathbb{R})$. Denote

$$\mathscr{C}^{n+\alpha}(\mathbb{R}) = \left\{ f \;\middle|\; \int_{-\infty}^{\infty} (1+|\kappa|)^{n+\alpha} |\hat{f}(\kappa)| d\kappa < \infty \right\},$$

where $\hat{f}(\kappa) = \int_{-\infty}^{\infty} e^{i\kappa x} f(x)dx$ is the Fourier transformation of $f(x)$.

Remark: A sufficient condition for $f \in \mathscr{C}^{n+\alpha}(\mathbb{R})$ is $f \in C^{n+2}(\mathbb{R})$ for $\alpha \in (0,1)$.

Lemma 2.3 [34] *Suppose that $f \in L_1(\mathbb{R})$ and $f \in \mathscr{C}^{1+\alpha}(\mathbb{R})$, and let*

$${}_{-\infty}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{-\infty}^{t} (t-\xi)^{-\alpha} f(\xi) d\xi$$

and

$$A_{\tau,r}^{\alpha} f(t) = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(t-(k-r)\tau), \tag{2.1}$$

where r *is an integer and* $g_k^{(\alpha)} = (-1)^k\binom{\alpha}{k}$. *Then*

$$A_{\tau,r}^{\alpha} f(t) = {}_{-\infty}D_t^{\alpha} f(t) + O(\tau)$$

uniformly in $t \in \mathbb{R}$ *as* $\tau \to 0$.

Remark: Here, ${}_{-\infty}D_t^{\alpha} f(t)$ is actually the α-th Riemann-Liouville fractional derivative of function $f(t)$.

In fact, the coefficients $g_k^{(\alpha)}$ $(0 < \alpha \leqslant 1)$ in (2.1) are the coefficients of the power series of the function $(1-z)^{\alpha}$, i.e.,

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}z^k = \sum_{k=0}^{\infty} g_k^{(\alpha)} z^k,$$

for all $-1 < z \leqslant 1$, and they can be evaluated recursively by

$$g_0^{(\alpha)} = 1, \quad g_k^{(\alpha)} = (1 - \frac{\alpha+1}{k})g_{k-1}^{(\alpha)}, \quad k = 1, 2, \ldots.$$

In addition, if $\alpha = 0$, we stipulate $g_0^{(\alpha)} = 1$ and $g_k^{(\alpha)} = 0$ when $k = 1, 2, \ldots$.

Lemma 2.4 [34] *The coefficients in* (2.1) *satisfy the following properties for* $0 < \alpha \leqslant 1$,

$$g_0^{(\alpha)} = 1, \quad g_1^{(\alpha)} = -\alpha \leqslant 0, \tag{2.2}$$

$$g_2^{(\alpha)} \leqslant g_3^{(\alpha)} \leqslant g_4^{(\alpha)} \leqslant \cdots \leqslant 0, \tag{2.3}$$

$$\sum_{k=0}^{\infty} g_k^{(\alpha)} = 0, \quad \sum_{k=0}^{n} g_k^{(\alpha)} \geqslant 0, \quad n \geqslant 1. \tag{2.4}$$

In [24], Tian et al. provided the following result.

Lemma 2.5 *Suppose that* $f \in L_1(\mathbb{R})$ *and* $f \in \mathscr{C}^{2+\alpha}(\mathbb{R})$. *We have*

$$(1+\frac{\alpha}{2})A_{\tau,0}^{\alpha} f(t) - \frac{\alpha}{2}A_{\tau,-1}^{\alpha} f(t) = {}_{-\infty}D_t^{\alpha} f(t) + O(\tau^2),$$

uniformly in $t \in \mathbb{R}$ *as* $\tau \to 0$.

We reformulate the left hand side of above equality as follows:

$$\begin{aligned}
&(1+\frac{\alpha}{2})A_{\tau,0}^{\alpha} f(t) - \frac{\alpha}{2}A_{\tau,-1}^{\alpha} f(t) \\
&= (1+\frac{\alpha}{2})\frac{1}{\tau^{\alpha}}\sum_{k=0}^{\infty} g_k^{(\alpha)} f(t-k\tau) - \frac{\alpha}{2}\frac{1}{\tau^{\alpha}}\sum_{k=0}^{\infty} g_k^{(\alpha)} f(t-(k+1)\tau) \\
&= (1+\frac{\alpha}{2})\frac{1}{\tau^{\alpha}}\sum_{k=0}^{\infty} g_k^{(\alpha)} f(t-k\tau) - \frac{\alpha}{2}\frac{1}{\tau^{\alpha}}\sum_{k=1}^{\infty} g_{k-1}^{(\alpha)} f(t-k\tau) \\
&= \frac{1}{\tau^{\alpha}}\left\{(1+\frac{\alpha}{2})g_0^{(\alpha)} f(t) + \sum_{k=1}^{\infty}\left[(1+\frac{\alpha}{2})g_k^{(\alpha)} - \frac{\alpha}{2}g_{k-1}^{(\alpha)}\right] f(t-k\tau)\right\}
\end{aligned}$$

$$= \frac{1}{\tau^\alpha} \sum_{k=0}^{\infty} \lambda_k^{(\alpha)} f(t - k\tau),$$

where

$$\lambda_0^{(\alpha)} = (1 + \frac{\alpha}{2}) g_0^{(\alpha)}; \quad \lambda_k^{(\alpha)} = (1 + \frac{\alpha}{2}) g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)}, \quad k \geqslant 1. \tag{2.5}$$

It can be checked for $0 \leqslant \alpha \leqslant 1$ that

$$\begin{aligned}
\lambda_0^{(\alpha)} &= 1 + \frac{\alpha}{2} > 0, \\
\lambda_1^{(\alpha)} &= -\frac{1}{2}(\alpha + 3)\alpha \leqslant 0, \\
\lambda_2^{(\alpha)} &= \frac{1}{4}(\alpha^2 + 3\alpha - 2)\alpha = \begin{cases} \leqslant 0, & \alpha \in [0, \frac{\sqrt{17}-3}{2}], \\ > 0, & \alpha \in (\frac{\sqrt{17}-3}{2}, 1], \end{cases} \\
\lambda_k^{(\alpha)} &= \left[(1 + \frac{\alpha}{2})(1 - \frac{1+\alpha}{k}) - \frac{\alpha}{2} \right] g_{k-1}^{(\alpha)} \leqslant 0, \quad k = 3, 4, 5, \ldots.
\end{aligned}$$

Lemma 2.6 [8] *Let* $\{\lambda_k^{(\alpha)}\}_{k=0}^{\infty}$ *be defined in (2.5). Then for any positive integer* m *and real vector* $(v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^m$, *it holds that*

$$\sum_{n=1}^{m} \left(\sum_{k=0}^{n-1} \lambda_k^{(\alpha)} v_{n-k} \right) v_n \geqslant 0.$$

Similarly, we can obtain

Lemma 2.7 *Let* $\{\lambda_k^{(\alpha)}\}_{k=0}^{\infty}$ *be defined as in (2.5). Then for any positive integer* m *and real vector* $(v_0, v_1, v_2, \ldots, v_m)^T \in \mathbb{R}^{m+1}$, *it holds that*

$$\sum_{n=0}^{m} \left(\sum_{k=0}^{n} \lambda_k^{(\alpha)} v_{n-k} \right) v_n \geqslant 0.$$

Denote

$$\mu = \triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \lambda_0^{(\alpha_l)}, \quad \nu = \triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \lambda_0^{(\alpha_l)}. \tag{2.6}$$

Lemma 2.8 *For* μ, ν *defined in (2.6), we have*

$$\mu = \frac{1}{O(\tau |\ln \tau|)}, \tag{2.7}$$

$$\nu = \frac{1}{O(\tau |\ln \tau|)}. \tag{2.8}$$

Proof.

$$\begin{aligned}
\mu &= \triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} (1 + \frac{\alpha_l}{2}) \\
&\sim \int_0^1 w(\alpha) \frac{1 + \frac{\alpha}{2}}{\tau^\alpha} d\alpha
\end{aligned}$$

$$= w(\alpha^*)(1+\frac{\alpha^*}{2})\int_0^1 (\frac{1}{\tau})^\alpha d\alpha$$

$$= w(\alpha^*)(1+\frac{\alpha^*}{2})\frac{(\frac{1}{\tau})^\alpha}{\ln\frac{1}{\tau}}\Bigg|_{\alpha=0}^{1}$$

$$= w(\alpha^*)(1+\frac{\alpha^*}{2})\frac{\frac{1}{\tau}-1}{\ln\frac{1}{\tau}},$$

which implies

$$\mu = \frac{1}{O(\tau|\ln\tau|)},$$

where $\alpha^* \in (0,1)$. Similarly, we can prove

$$\nu = \frac{1}{O(\tau|\ln\tau|)}.$$

This completes the proof. □

3 One-dimensional Problem

Consider the one-dimensional distributed-order problem

$$\mathcal{D}_t^w u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t), \quad 0 < x < L, \quad 0 < t \leqslant T, \tag{3.1}$$

$$u(x,0) = 0, \quad 0 \leqslant x \leqslant L, \tag{3.2}$$

$$u(0,t) = \psi_1(t), \quad u(L,t) = \psi_2(t), \quad 0 < t \leqslant T, \tag{3.3}$$

where $\psi_1(0) = 0,\ \psi_2(0) = 0$.

Divide the interval $[0, L]$ into M-subintervals with $h = \frac{L}{M}$ and $x_i = ih, i = 0, 1, 2, \ldots, M$.

Denote $\Omega_h = \{x_i \mid 0 \leqslant i \leqslant M\}$. Let $\mathcal{U}_h = \{v \mid v = (v_0, v_1, \ldots, v_M), v_0 = v_M = 0\}$ be the grid space on Ω_h. For any $u \in \mathcal{U}_h$ and $v \in \mathcal{U}_h$, introduce the following notations:

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}), \quad \mathcal{A}u_i = \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}),$$

$$(u,v) = h\sum_{i=1}^{M-1} u_i v_i, \quad (\delta_x u, \delta_x v) = h\sum_{i=1}^{M}(\delta_x u_{i-\frac{1}{2}})\delta_x v_{i-\frac{1}{2}}, \quad (\delta_x^2 u, \delta_x^2 v) = h\sum_{i=1}^{M-1}(\delta_x^2 u_i)\delta_x^2 v_i,$$

$$< \delta_x u, \delta_x v >= (\delta_x u, \delta_x v) - \frac{h^2}{12}(\delta_x^2 u, \delta_x^2 v), \quad \|u\| = \sqrt{(u,u)},$$

$$|u|_1 = \sqrt{(\delta_x u, \delta_x u)}, \quad |u|_{\mathcal{A}} = \sqrt{< \delta_x u, \delta_x u >}, \quad \|u\|_\infty = \max_{0\leqslant i\leqslant M}|u_i|.$$

Obviously, it holds that $< \delta_x u, \delta_x v >= -(\delta_x^2 u, \mathcal{A}v)$ for any $u, v \in \mathcal{U}_h$ and $|u|_{\mathcal{A}}^2 = -(\delta_x^2 u, \mathcal{A}u)$.

Two lemmas are given below that will be useful in the derivation and analysis of numerical schemes later.

Lemma 3.1 [30,35] *Let $u \in \mathcal{U}_h$. Then*

$$\|u\| \leqslant \frac{L}{\sqrt{6}}|u|_1,$$

$$\|u\|_\infty \leqslant \frac{\sqrt{L}}{2}|u|_1,$$
$$\frac{2}{3}|u|_1^2 \leqslant |u|_{\mathcal{A}}^2 \leqslant |u|_1^2,$$
$$\|\mathcal{A}u\| \leqslant \|u\|.$$

Lemma 3.2 [36, 37] *Let function* $g \in \mathcal{C}^6[x_{i-1}, x_{i+1}], x_{i+1} = x_i + h, x_{i-1} = x_i - h,$ *and* $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$, *then*

$$\frac{g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})}{12} = \frac{g(x_{i+1}) - 2g(x_i) + g(x_{i-1})}{h^2} + \frac{h^4}{360}\int_0^1 \left[g^{(6)}(x_i - sh) + g^{(6)}(x_i + sh)\right]\zeta(s)ds.$$

Denote

$$U_i^n = u(x_i, t_n), \quad F_i^n = F(x_i, t_n), \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant n \leqslant N.$$

3.1 A second-order method in space and distributed order

3.1.1 The derivation of the scheme

Considering (3.1) at the point (x_i, t_n), we have

$$\mathcal{D}_t^w u(x_i, t_n) = \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + F(x_i, t_n), \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N. \tag{3.4}$$

Let

$$s(\alpha, x_i, t_n) = w(\alpha)\, {}_0^C D_t^\alpha u(x_i, t_n).$$

Using Lemma 2.1, we arrive at

$$\begin{aligned}\mathcal{D}_t^w u(x_i, t_n) &= \int_0^1 s(\alpha, x_i, t_n)d\alpha \\ &= \triangle\alpha \sum_{l=0}^{2J} c_l s(\alpha_l, x_i, t_n) - \frac{\triangle\alpha^2}{12}\left.\frac{\partial^2 s(\alpha, x_i, t_n)}{\partial\alpha^2}\right|_{\alpha=\xi_i^n} \\ &= \triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\ {}_0^C D_t^{\alpha_l} u(x_i, t_n) + O(\triangle\alpha^2),\end{aligned} \tag{3.5}$$

where $\xi_i^n \in (0, 1)$.

Suppose $u(x,t) \in C^{(4,4)}([0,L]\times[0,T])$. In addition, suppose that $\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = 0$ $(k = 0, 1, \ldots, 4)$, which can be guaranteed in view of the Dimitrov's work [38]. Noticing the equivalence between the Riemann-Liouville fractional derivative ${}_{-\infty}D_t^\alpha f(t)$ with $f(t) = 0$ at $t \leqslant 0$ and the Caputo fractional derivative ${}_0^C D_t^\alpha f(t)$, by Lemma 2.5, we obtain

$$\mathcal{D}_t^w u(x_i, t_n) = \triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l)\left[\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} U_i^{n-k} + O(\tau^2)\right] + O(\triangle\alpha^2). \tag{3.6}$$

Substituting (3.6) into (3.4), one can obtain

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} U_i^{n-k} = \delta_x^2 U_i^n + F_i^n + p_i^n,$$

$$1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \tag{3.7}$$

where there exists a positive constant κ_1 such that

$$|p_i^n| \leqslant \kappa_1(\tau^2 + h^2 + \Delta\alpha^2), \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N. \tag{3.8}$$

Noticing the initial and boundary conditions (3.2)–(3.3), we have

$$U_i^0 = 0, \quad 0 \leqslant i \leqslant M, \tag{3.9}$$

$$U_0^n = \psi_1(t_n), \quad U_M^n = \psi_2(t_n), \quad 1 \leqslant n \leqslant N. \tag{3.10}$$

We construct the difference scheme for (3.1)–(3.3) as follows

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_i^{n-k} = \delta_x^2 u_i^n + F_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \tag{3.11}$$

$$u_i^0 = 0, \quad 0 \leqslant i \leqslant M, \tag{3.12}$$

$$u_0^n = \psi_1(t_n), \quad u_M^n = \psi_2(t_n), \quad 1 \leqslant n \leqslant N. \tag{3.13}$$

3.1.2 Stability

Theorem 3.1 *Let $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ be the solution of the following difference scheme*

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_i^{n-k} = \delta_x^2 u_i^n + F_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \tag{3.14}$$

$$u_i^0 = \phi_i, \quad 0 \leqslant i \leqslant M, \tag{3.15}$$

$$u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.16}$$

Then, we have

$$\tau \sum_{n=1}^{m} |u^n|_1^2 \leqslant 2\mu\tau \|u^0\|^2 + \frac{L^2}{6}\tau \sum_{n=1}^{m} \|F^n\|^2, \quad 1 \leqslant m \leqslant N,$$

where $\|F^n\|^2 = h \sum_{i=1}^{M-1} (F_i^n)^2$.

Proof. Making an inner product of (3.14) with u^n, using Cauchy-Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (u^{n-k}, u^n) \\
&= (\delta_x^2 u^n, u^n) + (F^n, u^n) \\
&= -|u^n|_1^2 + (F^n, u^n) \\
&\leqslant -|u^n|_1^2 + \frac{3}{L^2}\|u^n\|^2 + \frac{L^2}{12}\|F^n\|^2 \\
&\leqslant -|u^n|_1^2 + \frac{1}{2}|u^n|_1^2 + \frac{L^2}{12}\|F^n\|^2
\end{aligned}$$

$$= -\frac{1}{2}|u^n|_1^2 + \frac{L^2}{12}\|F^n\|^2, \quad 1 \leqslant n \leqslant N.$$

Summing up the above inequality for n from 1 to m, we obtain

$$\begin{aligned} &\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{n=1}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)}(u^{n-k}, u^n) \\ &\leqslant -\frac{1}{2}\sum_{n=1}^{m}|u^n|_1^2 + \frac{L^2}{12}\sum_{n=1}^{m}\|F^n\|^2, \quad 1 \leqslant m \leqslant N. \end{aligned}$$

Adding $\mu(u^0, u^0)$ on the both sides of above inequality, we have

$$\begin{aligned} &\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)}(u^{n-k}, u^n) \\ &\leqslant -\frac{1}{2}\sum_{n=1}^{m}|u^n|_1^2 + \mu(u^0, u^0) + \frac{L^2}{12}\sum_{n=1}^{m}\|F^n\|^2, \quad 1 \leqslant m \leqslant N. \end{aligned} \tag{3.17}$$

With the use of Lemma 2.7 we get

$$\sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)}(u^{n-k}, u^n) = \sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \cdot h \sum_{i=1}^{M-1} u_i^{n-k} u_i^n = h \sum_{i=1}^{M-1}\left[\sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_i^{n-k} u_i^n\right] \geqslant 0. \tag{3.18}$$

Combining (3.17) and (3.18) immediately arrives at

$$\tau \sum_{n=1}^{m} |u^n|_1^2 \leqslant 2\mu\tau\|u^0\|^2 + \frac{L^2}{6}\tau \sum_{n=1}^{m}\|F^n\|^2, \quad 1 \leqslant m \leqslant N.$$

This completes the proof. □

3.1.3 Convergence

Theorem 3.2 *Let $u(x,t)$ be the solution of the problem (3.1)–(3.3) and $\{u_i^n \mid 0 \leqslant i \leqslant M,\ 0 \leqslant n \leqslant N\}$ be the solution of difference scheme (3.11)–(3.13). Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant n \leqslant N.$$

Then we have

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{12} L^2 T \kappa_1 (\tau^2 + h^2 + \Delta\alpha^2).$$

Proof. Subtracting (3.11)–(3.13) from (3.7), (3.9)–(3.10), respectively, we obtain the error system of equations:

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} e_i^{n-k} = \delta_x^2 e_i^n + p_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \tag{3.19}$$

$$
\begin{aligned}
&e_i^0 = 0, \quad 0 \leqslant i \leqslant M, && (3.20)\\
&e_0^n = 0, \quad e_M^n = 0, \quad 1 \leqslant n \leqslant N. && (3.21)
\end{aligned}
$$

Applying Theorem 3.1 and (3.8), we get

$$
\begin{aligned}
\tau \sum_{n=1}^{N} |e^n|_1^2 &\leqslant \frac{L^2}{6} \tau \sum_{n=1}^{N} \|p^n\|^2 \\
&\leqslant \frac{L^2}{6} \tau \sum_{n=1}^{N} L \left[\kappa_1(\tau^2 + h^2 + \Delta\alpha^2)\right]^2 \leqslant \frac{L^3}{6} T \left[\kappa_1(\tau^2 + h^2 + \Delta\alpha^2)\right]^2. \qquad (3.22)
\end{aligned}
$$

By Cauchy-Schwarz inequality, Lemma 3.1 and (3.22), we have

$$
\begin{aligned}
&\left(\tau \sum_{n=1}^{N} \|e^n\|_\infty\right)^2 \leqslant \left(\tau \sum_{n=1}^{N} 1\right)\left(\tau \sum_{n=1}^{N} \|e^n\|_\infty^2\right) \\
&\leqslant T \cdot \frac{L}{4} \tau \sum_{n=1}^{N} |e^n|_1^2 \\
&\leqslant \frac{LT}{4} \cdot \frac{L^3}{6} T \left[\kappa_1(\tau^2 + h^2 + \Delta\alpha^2)\right]^2,
\end{aligned}
$$

or,

$$
\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{12} L^2 T \kappa_1(\tau^2 + h^2 + \Delta\alpha^2).
$$

This completes the proof. □

3.2 A fourth-order method in space and distributed order

3.2.1 The derivation of the scheme

Considering (3.1) at the point (x_i, t_n), we have

$$
\mathcal{D}_t^w u(x_i, t_n) = \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + F(x_i, t_n), \quad 0 \leqslant i \leqslant M, \quad 1 \leqslant n \leqslant N.
$$

Acting the operator $\mathcal{A}$ on the above equality leads to

$$
\mathcal{A}\mathcal{D}_t^w u(x_i, t_n) = \mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_n) + \mathcal{A}F_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N. \qquad (3.23)
$$

Let

$$
s(\alpha, x_i, t_n) = w(\alpha)\, {}_0^C D_t^\alpha u(x_i, t_n).
$$

Using Lemma 2.2, we arrive at

$$
\mathcal{D}_t^w u(x_i, t_n) = \int_0^1 s(\alpha, x_i, t_n) d\alpha
$$

$$= \triangle\alpha \sum_{l=0}^{2J} d_l s(\alpha_l, x_i, t_n) - \frac{\triangle\alpha^4}{180} \left.\frac{\partial^4 s(\alpha, x_i, t_n)}{\partial \alpha^4}\right|_{\alpha=\eta_i^n}$$
$$= \triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \ {}_0^C D_t^{\alpha_l} u(x_i, t_n) + O(\triangle\alpha^4),$$

where $\eta_i^n \in (0,1)$.

Suppose $u(x,t) \in C^{(6,4)}([0,L]\times[0,T])$ and $\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = 0$ $(k = 0,1,\dots,4)$. Noticing the equivalence between the Riemann-Liouville fractional derivative ${}_{-\infty}D_t^{\alpha} f(t)$ with $f(t)=0$ at $t \leqslant 0$ and the Caputo fractional derivative ${}_0^C D_t^{\alpha} f(t)$, by Lemma 2.5, we obtain

$$\mathcal{D}_t^w u(x_i, t_n) = \triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \left[\frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} U_i^{n-k} + O(\tau^2) \right] + O(\triangle\alpha^4). \quad (3.24)$$

Applying Lemma 3.2, we have

$$\mathcal{A}\frac{\partial^2 u}{\partial x^2}(x_i, t_n) = \delta_x^2 U_i^n + O(h^4). \quad (3.25)$$

Substituting (3.24) and (3.25) into (3.23), we obtain

$$\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \mathcal{A} U_i^{n-k} = \delta_x^2 U_i^n + \mathcal{A} F_i^n + q_i^n,$$
$$1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \quad (3.26)$$

where there exists a positive constant κ_2 such that

$$|q_i^n| \leqslant \kappa_2(\tau^2 + h^4 + \Delta\alpha^4), \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N. \quad (3.27)$$

Noticing the initial and boundary conditions (3.2)–(3.3), we have

$$U_i^0 = 0, \quad 0 \leqslant i \leqslant M, \quad (3.28)$$
$$U_0^n = \psi_1(t_n), \quad U_M^n = \psi_2(t_n), \quad 1 \leqslant n \leqslant N. \quad (3.29)$$

We construct the difference scheme for (3.1)–(3.3) as follows

$$\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \mathcal{A} u_i^{n-k} = \delta_x^2 u_i^n + \mathcal{A} F_i^n,$$
$$1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \quad (3.30)$$
$$u_i^0 = 0, \quad 0 \leqslant i \leqslant M, \quad (3.31)$$
$$u_0^n = \psi_1(t_n), \quad u_M^n = \psi_2(t_n), \quad 1 \leqslant n \leqslant N. \quad (3.32)$$

3.2.2 Stability

Theorem 3.3 *Let $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ be the solution of the following difference scheme*

$$\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \mathcal{A} u_i^{n-k} = \delta_x^2 u_i^n + \hat{F}_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \quad (3.33)$$

$$u_i^0 = \phi(x_i), \quad 0 \leqslant i \leqslant M, \tag{3.34}$$

$$u_0^n = 0, \quad u_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.35}$$

Then, we have

$$\tau \sum_{n=1}^{m} |u^n|_1^2 \leqslant 3\nu\tau \|u^0\|^2 + \frac{3L^2}{8}\tau \sum_{n=1}^{m} \|\hat{F}^n\|^2, \quad 1 \leqslant m \leqslant N.$$

Proof. Making an inner product of (3.33) with $\mathcal{A}u^n$, and using Cauchy-Schwarz inequality as well as Lemma 3.1, we have

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (\mathcal{A}u^{n-k}, \mathcal{A}u^n) \\
&= (\delta_x^2 u^n, \mathcal{A}u^n) + (\hat{F}^n, \mathcal{A}u^n) \\
&= -|u^n|_{\mathcal{A}}^2 + (\hat{F}^n, \mathcal{A}u^n) \\
&\leqslant -|u^n|_{\mathcal{A}}^2 + \|\hat{F}^n\| \cdot \|\mathcal{A}u^n\| \\
&\leqslant -|u^n|_{\mathcal{A}}^2 + \|\hat{F}^n\| \cdot \|u^n\| \\
&\leqslant -\frac{2}{3}|u^n|_1^2 + \frac{2}{L^2}\|u^n\|^2 + \frac{L^2}{8}\|\hat{F}^n\|^2 \\
&\leqslant -\frac{2}{3}|u^n|_1^2 + \frac{1}{3}|u^n|_1^2 + \frac{L^2}{8}\|\hat{F}^n\|^2 \\
&= -\frac{1}{3}|u^n|_1^2 + \frac{L^2}{8}\|\hat{F}^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned}$$

Summing up the above inequality for n from 1 to m, we obtain

$$\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=1}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (\mathcal{A}u^{n-k}, \mathcal{A}u^n) \right] \leqslant -\frac{1}{3} \sum_{n=1}^{m} |u^n|_1^2 + \frac{L^2}{8} \sum_{n=1}^{m} \|\hat{F}^n\|^2, \quad 1 \leqslant m \leqslant N. \tag{3.36}$$

Adding $\nu(\mathcal{A}u^0, \mathcal{A}u^0)$ on the both sides of (3.36) yields

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=0}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (\mathcal{A}u^{n-k}, \mathcal{A}u^n) \right] \\
&\leqslant -\frac{1}{3} \sum_{n=1}^{m} |u^n|_1^2 + \nu(\mathcal{A}u^0, \mathcal{A}u^0) + \frac{L^2}{8} \sum_{n=1}^{m} \|\hat{F}^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{3.37}$$

Using Lemma 2.7 and similarly to the derivation of (3.18), we have

$$\sum_{n=0}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (\mathcal{A}u^{n-k}, \mathcal{A}u^n) \geqslant 0. \tag{3.38}$$

It follows from (3.37), (3.38) and Lemma 3.1 that

$$\tau \sum_{n=1}^{m} |u^n|_1^2 \leqslant 3\nu\tau(\mathcal{A}u^0, \mathcal{A}u^0) + \frac{3L^2}{8}\tau \sum_{n=1}^{m} \|\hat{F}^n\|^2 \leqslant 3\nu\tau\|u^0\|^2 + \frac{3L^2}{8}\tau \sum_{n=1}^{m} \|\hat{F}^n\|^2, \quad 1 \leqslant m \leqslant N.$$

This completes the proof. □

3.2.3 Convergence

Theorem 3.4 *Let $u(x,t)$ be the solution of the problem (3.1)–(3.3) and $\{u_i^n \mid 0 \leqslant i \leqslant M, 0 \leqslant n \leqslant N\}$ be the solution of difference scheme (3.30)–(3.32). Denote*

$$e_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M, \quad 0 \leqslant n \leqslant N.$$

Then we have

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{8} L^2 T \kappa_2 (\tau^2 + h^4 + \Delta\alpha^4).$$

Proof. Subtracting (3.30)–(3.32) from (3.26), (3.28)–(3.29), respectively, we obtain the error system of equations:

$$\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \mathcal{A} e_i^{n-k} = \delta_x^2 e_i^n + q_i^n, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant n \leqslant N, \tag{3.39}$$

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant M, \tag{3.40}$$

$$e_0^n = 0, \quad e_M^n = 0, \quad 1 \leqslant n \leqslant N. \tag{3.41}$$

The application of Theorem 3.3 and (3.27) into (3.39)–(3.41) produces

$$\begin{aligned}
&\tau \sum_{n=1}^{N} |e^n|_1^2 \leqslant \frac{3L^2}{8} \tau \sum_{n=1}^{N} \|q^n\|^2 \\
&\leqslant \frac{3L^2}{8} \tau \sum_{n=1}^{N} L \left[\kappa_2(\tau^2 + h^4 + \Delta\alpha^4)\right]^2 \leqslant \frac{3L^3}{8} T \left[\kappa_2(\tau^2 + h^4 + \Delta\alpha^4)\right]^2.
\end{aligned} \tag{3.42}$$

By Cauchy-Schwarz inequality, Lemma 3.1 and (3.42), we have

$$\begin{aligned}
&\left(\tau \sum_{n=1}^{N} \|e^n\|_\infty\right)^2 \leqslant \left(\tau \sum_{n=1}^{N} 1\right) \left(\tau \sum_{n=1}^{N} \|e^n\|_\infty^2\right) \\
&\leqslant T \cdot \frac{L}{4} \tau \sum_{n=1}^{N} |e^n|_1^2 \\
&\leqslant \frac{LT}{4} \cdot \frac{3L^3}{8} T \left[\kappa_2(\tau^2 + h^4 + \Delta\alpha^4)\right]^2,
\end{aligned}$$

or,

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{\sqrt{6}}{8} L^2 T \kappa_2 (\tau^2 + h^4 + \Delta\alpha^4).$$

This completes the proof. □

4 Two-dimensional Problem

Consider the problem (1.1)–(1.3) in two-dimensional case as follows:

$$\mathcal{D}_t^w u(x,y,t) = \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) + G(x,y,t), \quad (x,y)\in\Omega, \quad t\in(0,T], \tag{4.1}$$

$$u(x,y,0) = 0, \quad (x,y)\in\bar{\Omega}, \tag{4.2}$$

$$u(x,y,t) = \psi(x,y,t), \quad (x,y)\in\partial\Omega, \quad t\in(0,T], \tag{4.3}$$

where $\Omega = (0,L_1)\times(0,L_2)$, $\psi(x,y,0)=0$ when $(x,y)\in\partial\Omega$.

Take two positive integers M_1 and M_2. Let $h_1 = L_1/M_1, h_2 = L_2/M_2, h = \max\{h_1,h_2\}$. Denote $x_i = ih_1, y_j = jh_2$, $\omega = \{(i,j) \mid 1\leqslant i\leqslant M_1-1, 1\leqslant j\leqslant M_2-1\}$, $\Upsilon = \{(i,j) \mid (x_i,y_j)\in\partial\Omega\}, \bar{\omega} = \omega\cup\Upsilon$.

Denote

$$\mathcal{V}_h = \big\{v \mid v = \{v_{ij} \mid 0\leqslant i\leqslant M_1, 0\leqslant j\leqslant M_2; v_{ij}=0 \text{ when } (i,j)\in\Upsilon\}\big\}.$$

Suppose $v,w\in\mathcal{V}_h$. Introduce the following notations:

$$\delta_x v_{i-\frac12,j} = \frac{1}{h_1}(v_{ij}-v_{i-1,j}), \qquad \delta_x^2 v_{ij} = \frac{1}{h_1}(\delta_x v_{i+\frac12,j} - \delta_x v_{i-\frac12,j}),$$

$$\delta_y v_{i,j-\frac12} = \frac{1}{h_2}(v_{ij}-v_{i,j-1}), \qquad \delta_y^2 v_{ij} = \frac{1}{h_2}(\delta_y v_{i,j+\frac12} - \delta_y v_{i,j-\frac12}), \quad \Delta_h v_{ij} = \delta_x^2 v_{ij} + \delta_y^2 v_{ij},$$

$$\mathcal{H}_1 v_{ij} = \frac{1}{12}(v_{i-1,j} + 10v_{ij} + v_{i+1,j}), \quad \mathcal{H}_2 v_{ij} = \frac{1}{12}(v_{i,j-1} + 10v_{ij} + v_{i,j+1}), \quad \mathcal{H}v_{ij} = \mathcal{H}_1\mathcal{H}_2 v_{ij},$$

$$(\delta_x v, \delta_x w) = h_1h_2\sum_{i=1}^{M_1}\sum_{j=1}^{M_2-1}(\delta_x v_{i-\frac12,j})\delta_x w_{i-\frac12,j}, \quad (\delta_y v, \delta_y w) = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2}(\delta_y v_{i,j-\frac12})\delta_y w_{i,j-\frac12},$$

$$(\delta_x^2 v, \delta_x^2 w) = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\delta_x^2 v_{ij})\delta_x^2 w_{ij}, \quad (\delta_y^2 v, \delta_y^2 w) = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\delta_y^2 v_{ij})\delta_y^2 w_{ij},$$

$$(v,w) = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} v_{ij}w_{ij}, \quad \|v\|_\infty = \max_{0\leqslant i\leqslant M_1, 0\leqslant j\leqslant M_2}|v_{ij}|,$$

$$\|\nabla_h v\|^2 = (\delta_x v, \delta_x v) + (\delta_y v, \delta_y v), \quad \|\Delta_h v\|^2 = h_1h_2\sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1}(\Delta_h v_{ij})^2.$$

Lemma 4.1 [36] *If $v\in\mathcal{V}_h$, then*

$$\frac13\|\nabla_h v\|^2 \leqslant (\mathcal{H}v, -\Delta_h v) \leqslant \|\nabla_h v\|^2.$$

Lemma 4.2 [36] *For any $v\in\mathcal{V}_h$, there exists a positive constant c such that*

$$\|v\|_\infty \leqslant c\|\Delta_h v\|.$$

Lemma 4.3 [36] *If $v\in\mathcal{V}_h$, then*

$$\frac23\|\Delta_h v\|^2 \leqslant \big(\mathcal{H}_2\delta_x^2 v + \mathcal{H}_1\delta_y^2 v, \Delta_h v\big) \leqslant \|\Delta_h v\|^2.$$

Denote

$$U_{ij}^n = u(x_i,y_j,t_n), \quad G_{ij}^n = G(x_i,y_j,t_n), \quad 0\leqslant i\leqslant M_1, \quad 0\leqslant j\leqslant M_2, \quad 0\leqslant n\leqslant N.$$

4.1 A second-order difference method in space and distributed order

4.1.1 The derivation of the scheme

Considering (4.1) at the point (x_i, y_j, t_n), we have

$$\mathcal{D}_t^w u(x_i, y_j, t_n) = \frac{\partial^2 u}{\partial x^2}(x_i, y_j, t_n) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j, t_n) + G_{ij}^n, \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \quad (4.4)$$

Suppose $u(x,y,t) \in C^{(4,4,4)}(\overline{\Omega} \times [0,T])$ and $\frac{\partial^k u(x,y,t)}{\partial t^k}|_{t=0} = 0$ $(k = 0, 1, \ldots, 4)$. Using Lemma 2.1 and Lemma 2.5, we obtain

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} U_{ij}^{n-k} = \Delta_h U_{ij}^n + G_{ij}^n + P_{ij}^n, \quad (i,j) \in \omega, \quad 1 \leqslant n \leqslant N, \quad (4.5)$$

where there exists a positive constant κ_3 such that

$$|P_{ij}^n| \leqslant \kappa_3(\tau^2 + h_1^2 + h_2^2 + \Delta\alpha^2), \quad (i,j) \in \omega,\ 1 \leqslant n \leqslant N. \quad (4.6)$$

Noticing the initial and boundary conditions (4.2)–(4.3), we have

$$U_{ij}^0 = 0, \qquad (i,j) \in \omega, \quad (4.7)$$
$$U_{ij}^n = \psi(x_i, y_j, t_n), \quad (i,j) \in \Upsilon, \quad 0 \leqslant n \leqslant N. \quad (4.8)$$

Thus, we construct the difference scheme for (4.1)–(4.3) as follows

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ij}^{n-k} = \Delta_h u_{ij}^n + G_{ij}^n, \quad (i,j) \in \omega, \quad 1 \leqslant n \leqslant N, \quad (4.9)$$
$$u_{ij}^0 = 0, \quad (i,j) \in \omega, \quad (4.10)$$
$$u_{ij}^n = \psi(x_i, y_j, t_n), \quad (i,j) \in \Upsilon, \quad 0 \leqslant n \leqslant N. \quad (4.11)$$

4.1.2 Stability

Theorem 4.1 *Let $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ be the solution of the following difference scheme*

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} u_{ij}^{n-k} = \Delta_h u_{ij}^n + G_{ij}^n, \quad (i,j) \in \omega, \quad 1 \leqslant n \leqslant N, \quad (4.12)$$
$$u_{ij}^0 = \phi_{ij}, \quad (i,j) \in \omega, \quad (4.13)$$
$$u_{ij}^n = 0, \quad (i,j) \in \Upsilon, \quad 0 \leqslant n \leqslant N. \quad (4.14)$$

Then, we have

$$\tau \sum_{n=1}^{m} \|\Delta_h u^n\|^2 \leqslant 2\mu\tau \|\nabla_h u^0\|^2 + \tau \sum_{n=1}^{m} \|G^n\|^2, \quad 1 \leqslant m \leqslant N.$$

Proof. Making an inner product of (4.12) with $-\Delta_h u^n$ and using Cauchy-Schwarz inequality, we have

$$\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (u^{n-k}, -\Delta_h u^n)$$

$$
\begin{aligned}
&= -(\Delta_h u^n, \Delta_h u^n) - (G^n, \Delta_h u^n) \\
&\leqslant -\|\Delta_h u^n\|^2 + \|G^n\| \cdot \|\Delta_h u^n\| \\
&\leqslant -\|\Delta_h u^n\|^2 + \frac{1}{2}\|\Delta_h u^n\|^2 + \frac{1}{2}\|G^n\|^2 \\
&= -\frac{1}{2}\|\Delta_h u^n\|^2 + \frac{1}{2}\|G^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned}
$$

Summing up the above inequality for n from 1 to m leads to

$$
\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=1}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (u^{n-k}, -\Delta_h u^n)\right] \\
&\leqslant -\frac{1}{2}\sum_{n=1}^{m} \|\Delta_h u^n\|^2 + \frac{1}{2}\sum_{n=1}^{m} \|G^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{4.15}
$$

Adding $\mu(u^0, -\Delta_h u^0)$ on the both sides of (4.15), we have

$$
\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} c_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (u^{n-k}, -\Delta_h u^n)\right] \\
&\leqslant -\frac{1}{2}\sum_{n=1}^{m} \|\Delta_h u^n\|^2 + \mu(u^0, -\Delta_h u^0) + \frac{1}{2}\sum_{n=1}^{m} \|G^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{4.16}
$$

Using Lemma 2.7 and noticing

$$
(u^{n-k}, -\Delta_h u^n) = (\delta_x u^{n-k}, \delta_x u^n) + (\delta_y u^{n-k}, \delta_y u^n),
$$

similarly to the derivation of (3.18), we can obtain

$$
\sum_{n=0}^{m}\sum_{k=0}^{n} \lambda_k^{(\alpha_l)} (u^{n-k}, -\Delta_h u^n) \geqslant 0. \tag{4.17}
$$

The combination of (4.16) and (4.17) yields

$$
\begin{aligned}
&\tau \sum_{n=1}^{m} \|\Delta_h u^n\|^2 \\
&\leqslant 2\mu\tau(u^0, -\Delta_h u^0) + \tau \sum_{n=1}^{m} \|G^n\|^2 \\
&= 2\mu\tau\|\nabla_h u^0\|^2 + \tau \sum_{n=1}^{m} \|G^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned}
$$

This completes the proof. □

4.1.3 Convergence

Theorem 4.2 *Let $u(x,y,t)$ be the solution of the problem (4.1)–(4.3) and $\{u_{ij}^n \mid (i,j) \in \bar{\omega}, 0 \leqslant n \leqslant N\}$ be the solution of difference scheme (4.9)–(4.11). Denote*

$$
e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j) \in \bar{\omega}, \quad 0 \leqslant n \leqslant N.
$$

Then we have

$$\tau\sum_{n=1}^{N}\|e^n\|_\infty \leqslant cT\sqrt{L_1L_2}\kappa_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2).$$

Proof. Subtracting (4.9)–(4.11) from (4.5), (4.7)–(4.8), respectively, we obtain the error system of equations:

$$\triangle\alpha\sum_{l=0}^{2J}c_lw(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}e_{ij}^{n-k}=\Delta_he_{ij}^n+P_{ij}^n,\quad (i,j)\in\omega,\quad 1\leqslant n\leqslant N, \tag{4.18}$$

$$e_{ij}^0=0,\quad (i,j)\in\omega, \tag{4.19}$$

$$e_{ij}^n=0,\quad (i,j)\in\Upsilon,\quad 0\leqslant n\leqslant N. \tag{4.20}$$

Starting from the error system (4.18)–(4.20) and using Theorem 4.1 together with (4.6), we immediately arrive at

$$\begin{aligned}&\tau\sum_{n=1}^{N}\|\Delta_he^n\|^2\leqslant\tau\sum_{n=1}^{N}\|P^n\|^2\\&\leqslant\tau\sum_{n=1}^{N}L_1L_2\left[\kappa_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)\right]^2\leqslant TL_1L_2\left[\kappa_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)\right]^2.\end{aligned} \tag{4.21}$$

Using Cauchy-Schwarz inequality, Lemma 4.2 and (4.21), we have

$$\begin{aligned}&\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty\right)^2\leqslant\left(\tau\sum_{n=1}^{N}1\right)\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)\\&\leqslant Tc^2\cdot\tau\sum_{n=1}^{N}\|\Delta_he^n\|^2\\&\leqslant c^2T^2L_1L_2\left[\kappa_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2)\right]^2,\end{aligned} \tag{4.22}$$

or,

$$\tau\sum_{n=1}^{N}\|e^n\|_\infty \leqslant cT\sqrt{L_1L_2}\kappa_3(\tau^2+h_1^2+h_2^2+\Delta\alpha^2).$$

This completes the proof. □

4.2 A fourth-order difference method in space and distributed order

4.2.1 The derivation of the scheme

Considering (4.1) at the point (x_i,y_j,t_n), we have

$$\mathcal{D}_t^wu(x_i,y_j,t_n)=\frac{\partial^2u}{\partial x^2}(x_i,y_j,t_n)+\frac{\partial^2u}{\partial y^2}(x_i,y_j,t_n)+G(x_i,y_j,t_n),\quad (i,j)\in\bar{\omega},\quad 1\leqslant n\leqslant N.$$

Acting the operator $\mathcal{H}$ on the above equality produces

$$\mathcal{H}\mathcal{D}_t^w u(x_i,y_j,t_n) = \mathcal{H}_2\left[\mathcal{H}_1\frac{\partial^2 u}{\partial x^2}(x_i,y_j,t_n)\right] + \mathcal{H}_1\left[\mathcal{H}_2\frac{\partial^2 u}{\partial y^2}(x_i,y_j,t_n)\right] + \mathcal{H}G_{ij}^n,$$
$$(i,j)\in\omega, \quad 1\leqslant n\leqslant N.$$

Suppose $u(x,y,t)\in C^{(6,6,4)}(\overline{\Omega}\times[0,T])$ and $\frac{\partial^k u(x,y,t)}{\partial t^k}|_{t=0}=0$ $(k=0,1,\ldots,4)$. Using Lemma 2.2, Lemma 2.5 and Lemma 3.2, we obtain

$$\triangle\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}\mathcal{H}U_{ij}^{n-k} = \mathcal{H}_2\delta_x^2 U_{ij}^n + \mathcal{H}_1\delta_y^2 U_{ij}^n + \mathcal{H}G_{ij}^n + Q_{ij}^n,$$
$$(i,j)\in\omega, \quad 1\leqslant n\leqslant N, \tag{4.23}$$

where there exists a positive constant κ_4 such that

$$|Q_{ij}^n|\leqslant\kappa_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4), \quad (i,j)\in\omega, \quad 1\leqslant n\leqslant N. \tag{4.24}$$

Noticing the initial and boundary conditions (4.2)–(4.3), we have

$$U_{ij}^0 = 0, \quad (i,j)\in\omega, \tag{4.25}$$
$$U_{ij}^n = \psi(x_i,y_j,t_n), \quad (i,j)\in\Upsilon, \quad 0\leqslant n\leqslant N. \tag{4.26}$$

Hence, the difference scheme for (4.1)–(4.3) can be derived as follows

$$\triangle\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}\mathcal{H}u_{ij}^{n-k} = \mathcal{H}_2\delta_x^2 u_{ij}^n + \mathcal{H}_1\delta_y^2 u_{ij}^n + \mathcal{H}G_{ij}^n,$$
$$(i,j)\in\omega, \quad 1\leqslant n\leqslant N, \tag{4.27}$$
$$u_{ij}^0 = 0, \quad (i,j)\in\omega, \tag{4.28}$$
$$u_{ij}^n = \psi(x_i,y_j,t_n), \quad (i,j)\in\Upsilon, \quad 0\leqslant n\leqslant N. \tag{4.29}$$

4.2.2 Stability

Theorem 4.3 *Let $\{u_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ be the solution of the following difference scheme*

$$\triangle\alpha\sum_{l=0}^{2J} d_l w(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}\mathcal{H}u_{ij}^{n-k} = \mathcal{H}_2\delta_x^2 u_{ij}^n + \mathcal{H}_1\delta_y^2 u_{ij}^n + \hat{G}_{ij}^n,$$
$$(i,j)\in\omega, 1\leqslant n\leqslant N, \tag{4.30}$$
$$u_{ij}^0 = \phi_{ij}, \quad (i,j)\in\omega, \tag{4.31}$$
$$u_{ij}^n = 0, \quad (i,j)\in\Upsilon, \quad 0\leqslant n\leqslant N. \tag{4.32}$$

Then, we have

$$\tau\sum_{n=1}^{m}\|\Delta_h u^n\|^2 \leqslant 3\nu\tau\|\nabla_h u^0\|^2 + \frac{9}{4}\tau\sum_{n=1}^{m}\|\hat{G}^n\|^2, \quad 1\leqslant m\leqslant N.$$

Proof. Taking an inner product of (4.30) with $-\Delta_h u^n$, using Lemma 4.3 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \left(\mathcal{H} u^{n-k}, -\Delta_h u^n\right) \\
=& -\left(\mathcal{H}_2 \delta_x^2 u^n + \mathcal{H}_1 \delta_y^2 u^n, \Delta_h u^n\right) - (\hat{G}^n, \Delta_h u^n) \\
\leqslant& -\frac{2}{3}\|\Delta_h u^n\|^2 - (\hat{G}^n, \Delta_h u^n) \\
\leqslant& -\frac{2}{3}\|\Delta_h u^n\|^2 + \frac{1}{3}\|\Delta_h u^n\|^2 + \frac{3}{4}\|\hat{G}^n\|^2 \\
=& -\frac{1}{3}\|\Delta_h u^n\|^2 + \frac{3}{4}\|\hat{G}^n\|^2, \quad 1 \leqslant n \leqslant N.
\end{aligned} \tag{4.33}$$

Summing up the above inequality (4.33) for n from 1 to m yields

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=1}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \left(\mathcal{H} u^{n-k}, -\Delta_h u^n\right)\right] \\
\leqslant& -\frac{1}{3}\sum_{n=1}^{m} \|\Delta_h u^n\|^2 + \frac{3}{4}\sum_{n=1}^{m} \|\hat{G}^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{4.34}$$

Adding $\nu\left(\mathcal{H} u^0, -\Delta_h u^0\right)$ on the both sides of (4.34), we obtain

$$\begin{aligned}
&\triangle\alpha \sum_{l=0}^{2J} d_l w(\alpha_l) \cdot \frac{1}{\tau^{\alpha_l}} \left[\sum_{n=0}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \left(\mathcal{H} u^{n-k}, -\Delta_h u^n\right)\right] \\
\leqslant& -\frac{1}{3}\sum_{n=1}^{m} \|\Delta_h u^n\|^2 + \nu\left(\mathcal{H} u^0, -\Delta_h u^0\right) + \frac{3}{4}\sum_{n=1}^{m} \|\hat{G}^n\|^2, \quad 1 \leqslant m \leqslant N.
\end{aligned} \tag{4.35}$$

Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are both symmetric and positive definite, there are two symmetric and positive definite $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{H}_1 = \mathcal{Q}_1^2$ and $\mathcal{H}_2 = \mathcal{Q}_2^2$. Therefore,

$$\begin{aligned}
&\left(\mathcal{H} u^{n-k}, -\Delta_h u^n\right) \\
=& \left(\mathcal{H}_1\mathcal{H}_2 u^{n-k}, -\delta_x^2 u^n - \delta_y^2 u^n\right) \\
=& \left(\mathcal{H}_1\mathcal{H}_2 u^{n-k}, -\delta_x^2 u^n\right) + \left(\mathcal{H}_1\mathcal{H}_2 u^{n-k}, -\delta_y^2 u^n\right) \\
=& \left(\mathcal{Q}_1\mathcal{Q}_2\delta_x u^{n-k}, \mathcal{Q}_1\mathcal{Q}_2\delta_x u^n\right) + \left(\mathcal{Q}_1\mathcal{Q}_2\delta_y u^{n-k}, \mathcal{Q}_1\mathcal{Q}_2\delta_y u^n\right).
\end{aligned} \tag{4.36}$$

Using Lemma 2.7 and noticing (4.36), similarly to the derivation of (3.18), we have

$$\sum_{n=0}^{m} \sum_{k=0}^{n} \lambda_k^{(\alpha_l)} \left(\mathcal{H} u^{n-k}, -\Delta_h u^n\right) \geqslant 0. \tag{4.37}$$

The application of (4.37) into (4.35) together with Lemma 4.1 achieves

$$\tau \sum_{n=1}^{m} \|\Delta_h u^n\|^2$$

$$\leqslant 3\nu\tau\left(\mathcal{H}u^0, -\Delta_h u^0\right) + \frac{9}{4}\tau\sum_{n=1}^{m}\|\hat{G}^n\|^2 \leqslant 3\nu\tau\|\nabla_h u^0\|^2 + \frac{9}{4}\tau\sum_{n=1}^{m}\|\hat{G}^n\|^2, \quad 1\leqslant m\leqslant N.$$

This completes the proof. □

4.2.3 Convergence

Theorem 4.4 *Let $u(x,y,t)$ be the solution of the problem (4.1)–(4.3) and $\{u_{ij}^n \mid (i,j)\in\bar{\omega}, 0\leqslant n\leqslant N\}$ be the solution of difference scheme (4.27)–(4.29). Denote*

$$e_{ij}^n = U_{ij}^n - u_{ij}^n, \quad (i,j)\in\bar{\omega}, \quad 0\leqslant n\leqslant N.$$

Then we have

$$\tau\sum_{n=1}^{N}\|e^n\|_\infty \leqslant \frac{3}{2}cT\sqrt{L_1L_2}\kappa_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4).$$

Proof. Subtracting (4.27)–(4.29) from (4.23), (4.25)–(4.26), respectively, we obtain the error system of equations:

$$\triangle\alpha\sum_{l=0}^{2J}d_l w(\alpha_l)\cdot\frac{1}{\tau^{\alpha_l}}\sum_{k=0}^{n}\lambda_k^{(\alpha_l)}\mathcal{H}e_{ij}^{n-k} = \mathcal{H}_2\delta_x^2 e_{ij}^n + \mathcal{H}_1\delta_y^2 e_{ij}^n + Q_{ij}^n,$$
$$(i,j)\in\omega, \quad 1\leqslant n\leqslant N, \tag{4.38}$$
$$e_{ij}^0 = 0, \quad (i,j)\in\omega, \tag{4.39}$$
$$e_{ij}^n = 0, \quad (i,j)\in\Upsilon, \quad 0\leqslant n\leqslant N. \tag{4.40}$$

Applying Theorem 4.3 and (4.24), we get

$$\begin{aligned}\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2 &\leqslant \frac{9}{4}\tau\sum_{n=1}^{N}\|Q^n\|^2\\ &\leqslant \frac{9}{4}\tau\sum_{n=1}^{N}L_1L_2\left[\kappa_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)\right]^2\\ &\leqslant \frac{9}{4}TL_1L_2\left[\kappa_4(\tau^2+h_1^4+h_2^4+\Delta\alpha^4)\right]^2.\end{aligned} \tag{4.41}$$

Using Cauchy-Schwarz inequality, Lemma 4.2 and (4.41), we have

$$\begin{aligned}\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty\right)^2 &\leqslant \left(\tau\sum_{n=1}^{N}1\right)\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)\\ &\leqslant T\left(\tau\sum_{n=1}^{N}\|e^n\|_\infty^2\right)\\ &\leqslant Tc^2\tau\sum_{n=1}^{N}\|\Delta_h e^n\|^2\end{aligned}$$

$$\leqslant Tc^2 \cdot \frac{9}{4} T L_1 L_2 \left[\kappa_4(\tau^2 + h_1^4 + h_2^4 + \Delta\alpha^4)\right]^2,$$

or,

$$\tau \sum_{n=1}^{N} \|e^n\|_\infty \leqslant \frac{3}{2} cT \sqrt{L_1 L_2} \kappa_4 (\tau^2 + h_1^4 + h_2^4 + \Delta\alpha^4).$$

This completes the proof. □

5 Numerical examples

In this section, several examples will be numerically computed to verify the efficiency and robustness of the proposed difference schemes.

5.1 One-dimensional case

Let $u^n = u^n(h, \tau, \Delta\alpha)$ be the solution of described difference scheme with the step sizes h, τ and $\Delta\alpha$. Denote

$$e(h, \tau, \Delta\alpha) = \max_{0 \leqslant n \leqslant N} \|U^n - u^n\|_\infty, \quad rate_\tau = \log_2 \frac{e(h, \tau, \Delta\alpha)}{e(h, \tau/2, \Delta\alpha)},$$

$$rate_h = \log_2 \frac{e(h, \tau, \Delta\alpha)}{e(h/2, \tau, \Delta\alpha)}, \quad rate_{\Delta\alpha} = \log_2 \frac{e(h, \tau, \Delta\alpha)}{e(h, \tau, \Delta\alpha/2)}.$$

Example 5.1 *In (3.1)–(3.3), take* $L = \pi, T = 0.5$, $w(\alpha) = \Gamma(4 - \alpha)$, $F(x, t) = 8[6(t^3 - t^2)/\ln(t) + t^3] \sin(x)$, $\psi_1(t) = \psi_2(t) = 0$.

The exact solution of the example is $u(x, t) = 8t^3 \sin(x)$.

We test the efficiency and numerical accuracy of difference scheme (3.11)–(3.13) and scheme (3.30)–(3.32) for computing this example, respectively. Firstly, the numerical accuracy in time variable is computed. Taking the fixed and sufficiently small h and $\Delta\alpha$, the numerical errors and convergence orders in discrete maximum norm are given in Table 1. The sufficiently small values of h and $\Delta\alpha$ can guarantee that the dominated errors arise from the approximation of temporal derivatives. For comparison, the numerical results of computing this example using the algorithm proposed in [21], denoted as *FMR-L1 scheme*, are also shown in the table. From Table 1, one can find that the second-order accuracy of these new developed two groups of difference schemes in time variable can both be obtained, while only a little more than first order accuracy is obtained for the *FMR-L1 scheme* in [21].

Secondly, the numerical accuracy of difference scheme (3.11)–(3.13) and scheme (3.30)–(3.32) in space variable is tested. Taking the fixed and sufficiently small τ and $\Delta\alpha$, the computational errors and numerical convergence orders in maximum norm are recorded in Table 2. From the table, the second-order convergence of difference scheme (3.11)–(3.13) and the fourth-order convergence of difference scheme (3.30)–(3.32) in space variable for computing this example, respectively, are verified.

Thirdly, we would like to investigate the numerical accuracy of difference scheme (3.11)–(3.13) and (3.30)–(3.32) in distributed-order variable. With the fixed and sufficiently large values of N and M, the computational results with different $\Delta\alpha$ are displayed in Table 3, from which, one can

Table 1: Maximum errors and convergence orders of difference schemes in time variable.

	scheme (3.11)–(3.13) ($M = 300, J = 100$)		scheme (3.30)–(3.32) ($M = 100, J = 100$)		*FMR-L1 scheme* ($M = 300, J = 100$)	
τ	$e(h, \tau, \Delta\alpha)$	$rate_\tau$	$e(h, \tau, \Delta\alpha)$	$rate_\tau$	$e(h, \tau, \Delta\alpha)$	$rate_\tau$
1/20	7.139630×10^{-3}	1.9659	7.140180×10^{-3}	1.9655	4.229500×10^{-2}	1.1698
1/40	1.827607×10^{-3}	1.9849	1.828195×10^{-3}	1.9835	1.879933×10^{-2}	1.1688
1/80	4.617229×10^{-4}	1.9975	4.623207×10^{-4}	1.9919	8.361821×10^{-3}	1.1616
1/160	1.156327×10^{-4}	2.0185	1.162330×10^{-4}	1.9960	3.737892×10^{-3}	1.1511
1/320	2.853888×10^{-5}	—	2.913973×10^{-5}	—	1.683125×10^{-3}	—

Table 2: Maximum errors and convergence orders of difference scheme (3.11)–(3.13) and (3.30)–(3.32) in space variable.

	scheme (3.11)–(3.13) ($N = 300, J = 50$)		scheme (3.30)–(3.32) ($N = 100000, J = 20$)	
h	$e(h, \tau, \Delta\alpha)$	$rate_h$	$e(h, \tau, \Delta\alpha)$	$rate_h$
$\pi/4$	4.840656×10^{-3}	1.9805	1.552013×10^{-4}	4.0250
$\pi/8$	1.226642×10^{-3}	1.9873	9.533232×10^{-6}	4.0103
$\pi/16$	3.093814×10^{-4}	1.9658	5.915861×10^{-7}	4.0637
$\pi/32$	7.920334×10^{-5}	—	3.537711×10^{-8}	—

Table 3: Maximum errors and convergence orders of difference scheme (3.11)–(3.13) and (3.30)–(3.32) in distributed-order variable.

	scheme (3.11)–(3.13) ($N = 2000, M = 200$)		scheme (3.30)–(3.32) ($N = 20000, M = 200$)	
$\Delta\alpha$	$e(h, \tau, \Delta\alpha)$	$rate_{\Delta\alpha}$	$e(h, \tau, \Delta\alpha)$	$rate_{\Delta\alpha}$
1/2	1.485739×10^{-2}	2.0087	2.960065×10^{-4}	3.9248
1/4	3.691933×10^{-3}	2.0047	1.949080×10^{-5}	3.9811
1/8	9.200055×10^{-4}	2.0107	1.234219×10^{-6}	4.0198
1/16	2.283002×10^{-4}	2.0416	7.608877×10^{-8}	4.4482
1/32	5.545448×10^{-5}	—	3.485687×10^{-9}	—

draw the conclusion that the convergence order of difference scheme (3.11)–(3.13) in distributed-order variable α is two, while it is four for difference scheme (3.30)–(3.32). The numerical results are in good agreement with the theoretical analysis.

In addition, the comparison between the difference scheme (3.11)–(3.13) and (3.30)–(3.32) will be shown. Taking $N = M = 2J$ for scheme (3.11)–(3.13) and $N = M^2 = (2J)^2$ for scheme (3.30)–(3.32), i.e. with an optimal step size ratio, the computational errors and CPU time on the same machine are recorded in Table 4, from which, one can find that, when the computational errors are on the same magnitudes, the compact difference scheme (3.30)–(3.32) can save much more CPU time than that of scheme (3.11)–(3.13).

Example 5.2 *In (3.1)–(3.3), take* $L = \pi, T = 0.5$, $w(\alpha) = \Gamma(\kappa + 1 - \alpha)$, $F(x,t) = 2^\kappa t^{\kappa-1} \cdot[\Gamma(\kappa+1)(t-1)/\ln(t) + t]\sin(x)$, $\psi_1(t) = \psi_2(t) = 0$, *where* κ *is a positive constant.*

The exact solution of the example is $u(x,t) = (2t)^\kappa \sin(x)$.

Table 4: Comparison between difference scheme (3.11)–(3.13) and (3.30)–(3.32).

	scheme (3.30)–(3.32)			scheme (3.11)–(3.13)		
N	$M=2J$	$e(h,\tau,\Delta\alpha)$	CPU (second)	$M=2J$	$e(h,\tau,\Delta\alpha)$	CPU (second)
256	16	1.007419×10^{-5}	0.0156	256	1.169473×10^{-5}	3.8844
400	20	4.143225×10^{-6}	0.0624	400	4.793061×10^{-6}	23.3534
576	24	1.998607×10^{-6}	0.1872	576	2.312230×10^{-6}	100.2462
784	28	1.078969×10^{-6}	0.5148	784	1.248330×10^{-6}	344.1226
1024	32	6.319707×10^{-7}	1.0140	1024	7.318413×10^{-7}	1003.9760

Maybe the readers have noticed that the condition $\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0}=0$ $(k=0,1,\ldots,4)$ is proposed during the construction of difference schemes to ensure the second-order accuracy in time variable. Is it necessary for the proposed algorithms? Maybe not since this condition is not completely satisfied in Example 5.1, for which, the ideal computational results can been obtained. To further uncover this confusion, in this example, we are concerned with the computational efficiency of both difference scheme (3.11)–(3.13) and scheme (3.30)–(3.32) for the cases $\kappa=2,3/2,1$, respectively. Table 5 lists the computational errors and convergence orders in time variable for these three cases, from which, one can find that the second-order convergence of both difference scheme (3.11)–(3.13) and scheme (3.30)–(3.32) in time variable can still be achieved for the case $\kappa=2$, whereas the numerical accuracy is obviously reduced to less than two for the latter two cases. Namely, certain conditions on the derivative values of function $u(x,t)$ at $t=0$ up to a necessary order are essential to ensure the second-order convergence of both difference scheme (3.11)–(3.13) and scheme (3.30)–(3.32) in time variable, whereas the condition $\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0}=0$ $(k=0,1,\ldots,4)$ is only sufficient but not necessary.

Table 5: Maximum errors and convergence orders of difference scheme (3.11)–(3.13) and (3.30)–(3.32) in time variable.

		scheme (3.11)–(3.13) $(M=300, J=100)$		scheme (3.30)–(3.32) $(M=100, J=100)$	
	τ	$e(h,\tau,\Delta\alpha)$	$rate_\tau$	$e(h,\tau,\Delta\alpha)$	$rate_\tau$
$\kappa=2$	1/20	2.604194×10^{-3}	1.8104	2.604625×10^{-3}	1.8103
	1/40	7.424869×10^{-4}	1.8622	7.426864×10^{-4}	1.8621
	1/80	2.042197×10^{-4}	1.8775	2.042987×10^{-4}	1.8767
	1/160	5.557797×10^{-5}	1.9057	5.563078×10^{-5}	1.9053
	1/320	1.483300×10^{-5}	—	1.485131×10^{-5}	—
$\kappa=3/2$	1/20	1.084362×10^{-3}	1.5048	1.084956×10^{-3}	1.5043
	1/40	3.821180×10^{-4}	1.1474	3.824538×10^{-4}	1.1555
	1/80	1.725077×10^{-4}	1.1742	1.716824×10^{-4}	1.1777
	1/160	7.644502×10^{-5}	1.2301	7.589344×10^{-5}	1.2346
	1/320	3.258713×10^{-5}	—	3.225094×10^{-5}	—
$\kappa=1$	1/20	1.864457×10^{-2}	0.7776	1.864251×10^{-2}	0.7778
	1/40	1.087577×10^{-2}	0.8112	1.087337×10^{-2}	0.8113
	1/80	6.198339×10^{-3}	0.8639	6.196568×10^{-3}	0.8640
	1/160	3.405682×10^{-3}	0.9007	3.404474×10^{-3}	0.9008
	1/320	1.824234×10^{-3}	—	1.823451×10^{-3}	—

5.2 Two-dimensional case

For simplicity, take $h_1 = h_2 = \widehat{h}$. Let $u^n = u^n(\widehat{h}, \tau, \Delta\alpha)$ be the solution of described difference schemes with the step sizes $\widehat{h}$, τ and $\Delta\alpha$. Denote

$$E(\widehat{h}, \tau, \Delta\alpha) = \max_{0 \leqslant n \leqslant N} \|U^n - u^n\|_\infty, \quad \widehat{rate}_\tau = \log_2 \frac{E(\widehat{h}, \tau, \Delta\alpha)}{E(\widehat{h}, \tau/2, \Delta\alpha)},$$

$$\widehat{rate}_1 = \log_2 \frac{E(\widehat{h}, \tau, \Delta\alpha)}{E(\widehat{h}/2, \tau/2, \Delta\alpha/2)}, \quad \widehat{rate}_2 = \log_2 \frac{E(\widehat{h}, \tau, \Delta\alpha)}{E(\widehat{h}/2, \tau/4, \Delta\alpha/2)}.$$

Example 5.3 *In (4.1)–(4.3), take* $L_1 = L_2 = \pi, T = 0.5$, $w(\alpha) = \Gamma(4-\alpha)$, $G(x, y, t) = 16[3(t^3 - t^2)/\ln(t) + t^3]\sin(x+y)$, $\psi(x, y, t) = t^3 \sin(x+y)$.

The analytic solution of the example is given by $u(x, y, t) = 8t^3 \sin(x+y)$.

We use the difference scheme (4.9)–(4.11) and scheme (4.27)–(4.29) to compute this example, respectively. Let $M_1 = M_2 = \widehat{M}$. Firstly, the numerical convergence orders of these two groups of difference schemes in time variable are verified. Taking the fixed and sufficiently small $\widehat{h}$ and $\Delta\alpha$, the computational errors and numerical convergence orders in discrete maximum norm are given in Table 6 with different temporal step sizes, from which, the second-order convergence of both difference scheme (4.9)–(4.11) and scheme (4.27)–(4.29) in time variable is apparent.

Table 6: Maximum errors and convergence orders of difference scheme (4.9)–(4.11) and (4.27)–(4.29) in time variable.

	scheme (4.9)–(4.11) ($\widehat{M} = 400, J = 100$)		scheme (4.27)–(4.29) ($\widehat{M} = 100, J = 50$)	
τ	$E(\widehat{h}, \tau, \Delta\alpha)$	$\widehat{rate}_\tau$	$E(\widehat{h}, \tau, \Delta\alpha)$	$\widehat{rate}_\tau$
1/20	5.104877×10^{-3}	1.9755	5.112379×10^{-3}	1.9739
1/40	1.298053×10^{-3}	1.9848	1.301407×10^{-3}	1.9878
1/80	3.279523×10^{-4}	1.9974	3.281217×10^{-4}	1.9938
1/160	8.213349×10^{-5}	2.0112	8.238147×10^{-5}	1.9969
1/320	2.037523×10^{-5}	—	2.063935×10^{-5}	

Secondly, with an optimal step size ratio, the example is computed using difference schemes (4.9)–(4.11) and (4.27)–(4.29), respectively. Taking $N = \widehat{M} = 2J$ for difference scheme (4.9)–(4.11) and $N = \widehat{M}^2 = (2J)^2$ for the other one, the computational errors and convergence orders are displayed in Table 7. One can see from the table that, when the step sizes in both spatial variable and distributed-order variable are reduced by a factor of 2, the errors are decreased approximately by a factor of 4 and 16, respectively. Hence, the convergence orders of difference scheme (4.9)–(4.11) in both spatial variable and distributed order variable are two, while that of difference scheme (4.27)–(4.29) is four. The close agreement of numerical results with theoretical analysis is confirmed.

6 Conclusion

In this paper, we have dealt with the numerical solutions to a class of distributed-order differential equations. Several effective difference schemes are developed for one-dimensional and two-dimensional problems. The unconditional stability and convergence of the obtained schemes

Table 7: Maximum errors and convergence orders of difference scheme (4.9)–(4.11) and (4.27)–(4.29).

	scheme (4.9)–(4.11)			scheme (4.27)–(4.29)		
$\widehat{M}=2J$	N	$E(\widehat{h},\tau,\Delta\alpha)$	$\widehat{rate_1}$	N	$E(\widehat{h},\tau,\Delta\alpha)$	$\widehat{rate_2}$
10	10	5.844001×10^{-3}	1.9554	100	5.807572×10^{-5}	3.9836
20	20	1.506847×10^{-3}	1.9781	400	3.671353×10^{-6}	3.9396
40	40	3.824717×10^{-4}	1.9942	1600	2.392735×10^{-7}	4.0756
80	80	9.600241×10^{-5}	—	6400	1.419155×10^{-8}	—

are investigated using the energy method. The global second-order convergence of the obtained schemes in time variable is attainable. In future work, the alternating direction implicit schemes will be considered for two-dimensional distributed-order differential equation. And the high order numerical scheme for the time-space fractional differential equation of distributed order will be investigated.

Acknowledgements

The research is supported by the research grant 11271068, 11326225, 11401319 from National Natural Science Foundation of China, BK20130860 from Natural Science Youth Foundation of Jiangsu Province, NY213051 from the Scientific Research Foundation of Nanjing University of Posts and Telecommunications, 105/2012/A3 from FDCT of Macao, and MYRG102(Y2-L3)-FST13-SHW from University of Macau.

References

[1] S. Yuste and L. Acedo, *An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations*, SIAM J. Numer. Anal. 42 (2005), pp. 1862–1874.

[2] T. Langlands and B. Henry, *The accuracy and stability of an implicit solution method for the fractional diffusion equation*, J. Comput. Phys. 205 (2005), pp. 719–736.

[3] C. Chen, F. Liu, I. Turner and V. Anh, *A Fourier method for the fractional diffusion equation describing sub-diffusion*, J. Comput. Phys. 227 (2007), pp. 886–897.

[4] P. Zhuang, F. Liu, V. Anh and I. Turner, *New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation*, SIAM J. Numer. Anal. 46 (2008), pp. 1079–1095.

[5] J. Cao and C. Xu, *A high order schema for the numerical solution of the fractional ordinary differential equations*, J. Comput. Phys. 238 (2013), pp. 154–168.

[6] C. Li and H. Ding, *Higher order finite difference method for the reaction and anomalous-diffusion equation*, Appl. Math. Model. (2014), doi:10.1016/j.apm.2013.12.002.

[7] Z.Z. Sun and X. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math. 56 (2006), pp. 193–209.

[8] Z. Wang and S. Vong, *Compact difference schemes for the modified anomalous fractional sub-diffusion equation and the fractional diffusion-wave equation*, J. Comput. Phys. 277 (2014), pp. 1–15.

[9] C. Li and F. Zeng, *Finite difference methods for fractional differential equations*, Int. J. Bifurcation Chaos 22 (2012), doi: 10.1142/S0218127412300145

[10] I.M. Sokolov, A.V. Checkin and J. Klafter, *Distributed-order fractional kinetics*, Acta Phys. Pol. A 35 (2004), pp. 1323–1341.

[11] K. Diethelm and N.J. Ford, *Numerical analysis for distributed order differential equations*, J. Comp. Appl. Math. 225 (2009), pp. 96–104.

[12] M. Caputo, *Elasticità e dissipazione*, Zanichelli Publisher, Bologna, 1969.

[13] M. Caputo, *Mean fractional-order-derivatives differential equations and filters*, AnnaliUniv. Ferrara, Sez. VII, Sci. Matemat. 41 (1995), pp. 73–83.

[14] I. Podlubny, T. Skovranek, B.M. Jara, I. Petras, V. Verbitsky and Y.Q. Chen, *Matrix approach to discrete fractional calculus III: non-equidistant grids, variable step length and distributed orders*, Phil. Trans. R. Soc. A 371: 20120153. http://dx.doi.org/10.1098/rsta.2012.0153

[15] Z. Jiao, Y.Q. Chen and I. Podlubny, *Distributed-order Dynamic Systems*, Berlin, Germany, Springer, 2012.

[16] Y. Luchko, *Boundary value problems for the generalized time-fractional diffusion equation of distributed order*, Fract. Calc. Appl. Anal. 12 (2009), pp. 409–422.

[17] R. Gorenflo, Y. Luchko and M. Stojanović, *Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density*, Fract. Calc. Appl. Anal. 16 (2013), pp. 297–316.

[18] F. Liu, M.M. Meerschaert, R.J. McGough, P. Zhuang and Q. Liu, *Numerical methods for solving the multi-term time-fractional wave-diffusion equation*, Fract. Calc. Appl. Anal. 16 (2013), pp. 9–25.

[19] H. Ye, F. Liu, V. Anh and I. Turner, *Numerical analysis for the time distributed-order and Riesz space fractional diffusions on bounded domains*, Int. J. Appl. Math. (2014), doi: 10.1093/imamat/hxu015

[20] H. Ye, F. Liu and V. Anh, *Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains*, J. Comput. Phys. , 2014, in press.

[21] N.J. Ford, M.L. Morgado and M. Rebelo, *A numerical method for the distributed order time-fractional diffusion equation*, ICFDA'14 Catania, 23-25 June 2014, ISBN 978-1-4799-2590-2.

[22] M.L. Morgado and M. Rebelo, *Numerical approximation of distributed order reaction-diffusion equations*, J. Comput. Appl. Math. (2014), http://dx.doi.org/10.1016/j.cam.2014.07.029

[23] J.T. Katsikadelis, *Numerical solution of distributed order fractional differential equations*, J. Comput. Phys. 259 (2014), pp. 11–22.

[24] W.Y. Tian, H. Zhou and W.H. Deng, *A class of second order difference approximations for solving space fractional diffusion equations*, Math. Comp. (2014), To appear.

[25] M. Cui, *Compact finite difference method for the fractional diffusion equation*, J. Comput. Phys. 228 (2009), pp. 7792–7804.

[26] M. Cui, *Convergence analysis of high-order compact alternating direction implicit schemes for the two-dimensional time fractional diffusion equation*, Numer. Algor. 62 (2013), pp. 383–409.

[27] C. Chen, F. Liu, I. Turner and V. Anh, *Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation*, SIAM J. Sci. Comput. 32 (2010), pp. 1740–1760.

[28] X. Hu and L. Zhang, *Implicit compact difference schemes for the fractional cable equation*, Appl. Math. Model. 36 (2012), pp. 4027–4043.

[29] X. Hu and L. Zhang, *A compact finite difference scheme for the fourth-order fractional diffusion-wave system*, Comput. Phys. Comm. 182 (2011), pp. 1645–1650.

[30] G.H. Gao and Z.Z. Sun, *A compact finite difference scheme for the fractional sub-diffusion equations*, J. Comput. Phys. 230 (2011), pp. 586–595.

[31] J. Ren and Z.Z. Sun, *Numerical algorithm with high spatial accuracy for the fractional diffusion-wave equation with Neumann boundary conditions*, J. Sci. Comput. 56 (2013), pp. 381–408.

[32] J. Ren, Z.Z. Sun and X. Zhao, *Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions*, J. Comput. Phys. 232 (2013), pp. 456–467.

[33] Y. Zhang, Z.Z. Sun and X. Zhao, *Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation*, SIAM J. Numer. Anal. 50 (2012), pp. 1535–1555.

[34] M.M. Meerschaert and C. Tadjeran, *Finite difference approximations for fractional advection-dispersion equations*, J. Comput. Appl. Math. 172 (2004), pp. 65–77.

[35] Z.Z. Sun, *Numerical Methods of Partial Differential Equations*, 2nd edition, Science Press, Beijing, 2012 (in Chinese).

[36] H.L. Liao and Z.Z. Sun, *Maximum norm error bounds of ADI and compact ADI methods for solving parabolic equations*, Numer. Methods Partial Differential Eq. 26 (2010), pp. 37–60.

[37] Z.Z. Sun, *The Method of Order Reduction and Its Application to the Numerical Solutions of Partial Differential Equations*, Science Press, Beijing, 2009.

[38] Y. Dimitrov, *Numerical approximation for fractional differential equations*, arXiv:1311.3935v1 [math.NA]15 Nov 2013