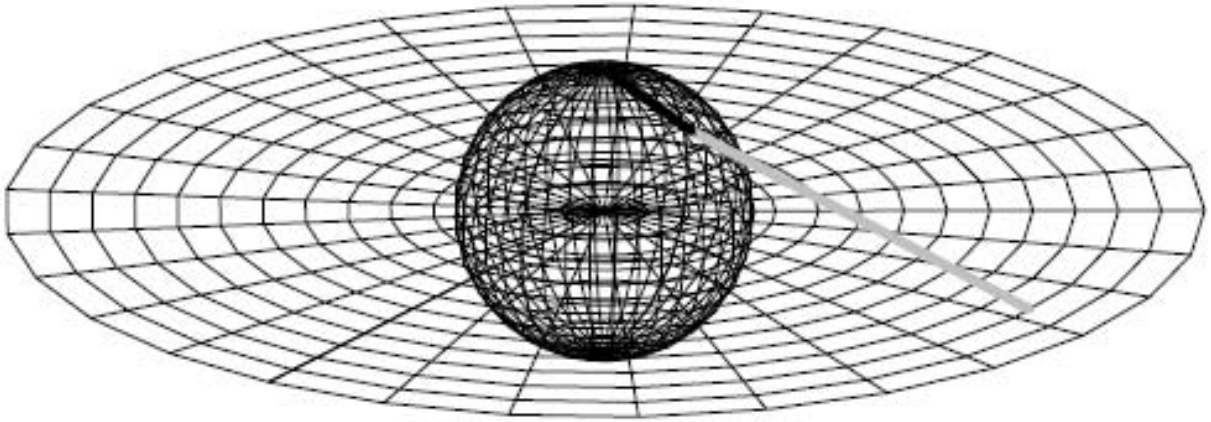
Complex vs Real: Stereographic Projection



Inversion

Reference: Toth G. Glimpses of Algebra and Geometry
(UTM, Springer-Verlag, 2002)

Youtbue Link to Möbius Transformation

1. This is the best to illustrate the Möbius transformation in terms of stereographic projection:
   http://www.youtube.com/watch?v=0z1fIsUNhO4
2. After watching the first one, and watch the 2nd one:
   `http://www.youtube.com/watch?v=sSfXMOmDKvM`
3. Angel and Devil:
   `http://www.youtube.com/watch?v=N3ttQJyLTSs`
4. Math and Arts (Póincare Kaleidoscope 1) 1:19 -1:20
   `http://www.youtube.com/watch?v=VlORNp3JNGA`
5. Circull Touch:
   `http://www.youtube.com/watch?v=V_7sgAX3u_M`

Warning. You must pay attention to the following identification, which is completely new to most of you.

Example. A complex number $z = a + bi \in \mathbb{C}, \quad a, b \in \mathbb{R}$, can be think as a 3-vector $(a, b, 0) \in \mathbb{R}^3$, i.e.

$$a + bi \in \mathbb{C} \leftrightsquigarrow (a, b, 0) \in \mathbb{R}^3.$$

Let's try your understanding now :') ! : }

1. What is the complex number given by $(0, 1, 0) =$ ____ $\in \mathbb{C}$?
   Answer. It is $\sqrt{-1} = i$.
2. What is the 3-vector representing
   $$\frac{1}{a+ib} = \left(\frac{a}{a^2+b^2}, \ -\frac{b}{a^2+b^2}, \ 0 \right) \in \mathbb{R}^3.$$

Answer. $$\frac{1}{a+ib} = \frac{(a-ib)}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}.$$

Example. A complex number $z = a + bi \in \mathbb{C}, \quad a, b \in \mathbb{R}$, can be think as a 3-vector $(a, b, 0) \in \mathbb{R}^3$, i.e. $a + bi \in \mathbb{C} \leftrightsquigarrow (a, b, 0) \in \mathbb{R}^3$.

In this case, one can think of the complex plane $\mathbb{C} = \{\ x + iy \mid x, y \in \mathbb{R}\ \}$ placed in the xy-plane inside the 3-dimensional Euclidean plane.

Remark. Note that $\mathbb{C}$ has a multiplication, namely $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$, but in general there is no such a multiplication in $\mathbb{R}^n$, except few special values of n.

If $n = 4$, there is a multiplication of vectors in $\mathbb{R}^4$ given by the cross-product $\times$ as follows:
$(a, b, c, d) \in \mathbb{R}^4 \leftrightsquigarrow a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{R}^1 \oplus \mathbb{R}^3 \cong \mathbb{R}^4$. Then
$(a, b, c, d) \cdot (p, q, r, s) = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \times (p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k})$
$= ap + (aq + bp + cs - dr)\mathbf{i} + (ar + cp + dq - bs)\mathbf{j}$
$\quad + (as + dp + br - cq)\mathbf{k}.$

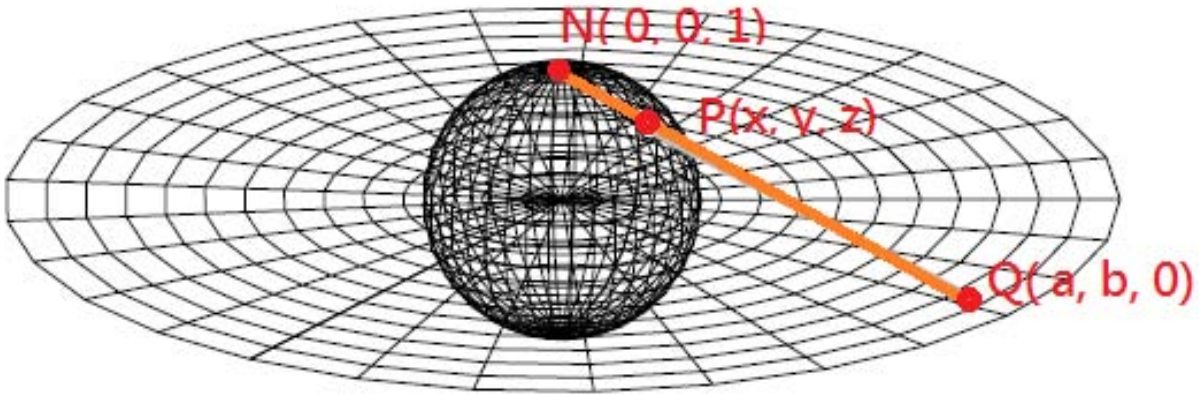
Sphere. Let $\mathbb{S}^2$ be the 2-sphere in $\mathbb{R}^3$ as $\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}$, i.e. all the locus of points $P(x, y, z)$ in $\mathbb{R}^3$, such that $\sqrt{x^2 + y^2 + z^2} = \|OP\| = 1$. In particular, $\mathbb{S}^2$ intersects the complex plane $\mathbb{C}$ $(= \mathbb{R}^2)$ inside $\mathbb{R}^3$ on unit circle $\mathbb{S}^1 = \{ z = x + iy = (x, y) \in \mathbb{C} \mid |z| = \sqrt{x^2 + y^2} = 1 \}$.

As $\mathbb{S}^2$ is a sphere, the highest point $N(0, 0, 1)$ (in terms of z-coordinates) will be called the north pole of $\mathbb{S}^2$, just like the one on Earth.

Remark. The integer n appeared in $\mathbb{S}^n$ represents the dimension of the sphere

$$\mathbb{S}^n = \{ (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \}$$

inside $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.



Define stereographic map $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ as follows: For a point $P(x, y, z) \in \mathbb{S}^2 \setminus \{N\}$, consider a ray $\overrightarrow{NP}$ from N to P, it intersects the complex plane $\mathbb{C}$ at a point $h_N(P) = Q$. Denote the coordinates of Q by $a + ib \in \mathbb{C}$, i.e. $Q(a, b, 0)$ in $\mathbb{R}^3$, then we can describe the map h_N in terms coordinates of $P(x, y, z)$.

Observe that points N, P and Q are collinear, i.e. in vector form $\overrightarrow{OQ} = \overrightarrow{ON} + \overrightarrow{NQ} = \overrightarrow{ON} + \lambda\overrightarrow{NP}$ for some positive λ. In terms of coordinates, $(a, b, 0) = (0, 0, 1) + \lambda(x - 0, y - 0, z - 1)$, $\heartsuit$
hence $1 + \lambda(z - 1) = 0$, i.e. $\lambda = \frac{1}{1-z}$. Note $z \neq 1$ as $P \neq N$. It follows from $\lambda x = a$, and $\lambda y = b$, that

$$h_N(x, y, z) = (a, b, 0) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right).$$

Properties of Stereographic Projection

In this case, $h_N(0,0,-1) = (0,0)$. We will prove

Proposition. $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is a bijective map.

Proof. h_N is injective: Suppose that two points (x,y,z), (x',y',z') in the unit 2-sphere have the same image by h_N, then
$(\frac{x}{1-z}, \frac{y}{1-z}, 0) = h_N(x,y,z) = h_N(x',y',z') = (\frac{x'}{1-z'}, \frac{y'}{1-z'}, 0)$.
In particular, $\dfrac{x}{1-z} = \dfrac{x'}{1-z'} \quad (1)$ and $\dfrac{y}{1-z} = \dfrac{y'}{1-z'} \quad (2)$.

Using $x^2+y^2+z^2 = 1 = x'^2+y'^2+z'^2$ in $(1)^2+(2)^2$, we have

$$\frac{x^2+y^2}{(1-z)^2} = \frac{(x')^2+(y')^2}{(1-z')^2} = \frac{1-(z')^2}{(1-z')^2} = \frac{1+z'}{1-z'}$$

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In particular, $\dfrac{x}{1-z} = \dfrac{x'}{1-z'} \quad (1)$ and $\dfrac{y}{1-z} = \dfrac{y'}{1-z'} \quad (2)$.

Using $x^2+y^2+z^2 = 1 = x'^2+y'^2+z'^2$ in $(1)^2+(2)^2$, we have
$\dfrac{1+z}{1-z} = \dfrac{1-z^2}{(1-z)^2} = \dfrac{x^2+y^2}{(1-z)^2} = \dfrac{(x')^2+(y')^2}{(1-z')^2} = \dfrac{1-(z')^2}{(1-z')^2} = \dfrac{1+z'}{1-z'}$, so
$(1-z')(1+z) = (1-z)(1+z')$ i.e. $z = z'$, and hence $x = x'$ and $y = y'$.

Proposition. $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is a bijective map.

Proof. h_N is surjective: We first give some discussion first before the proof. Geometrically, this property is clear. But one still needs a proof.
For any $a + ib \in \mathbb{C}$ (or $(a, b, 0) \in \mathbb{R}^3$), we want to find the inverse image (x, y, z) of $a + ib \in \mathbb{C}$ under h_N, i.e. to solve the unknown point $(x, y, z) \in \mathbb{S}^2$ in the equation $(a, b, 0) = h_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right)$.
The answer is

$$P(x, y, z) = \left(\frac{2a}{1 + a^2 + b^2}, \frac{2b}{1 + a^2 + b^2}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right).$$

In the following, we will only check the point $P(x, y, z)$ is indeed a solution.

Question. Could you find out the solution $P(x, y, z)$ above for given $a + ib \in \mathbb{C}$?

Proposition. $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is a bijective map.

Proof. h_N is surjective: For any $a + ib \in \mathbb{C}$ (or $(a, b, 0) \in \mathbb{R}^3$),

let $P(x, y, z) = \left(\dfrac{2a}{1 + a^2 + b^2}, \dfrac{2b}{1 + a^2 + b^2}, \dfrac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right)$. Then

$$x^2 + y^2 + z^2 = \frac{4a^2 + 4b^2 + (a^2 + b^2 - 1)^2}{(1 + a^2 + b^2)^2}$$

$$= \frac{(a^2 + b^2)^2 + 2(a^2 + b^2) + 1}{(1 + a^2 + b^2)^2} = 1, \text{ so } P \in \mathbb{S}^2.$$

Recall that $h_N(x, y, z) = \left(\dfrac{x}{1 - z}, \dfrac{y}{1 - z}, 0 \right)$, $\spadesuit$

and $1 - z = \dfrac{a^2 + b^2 + 1}{a^2 + b^2 + 1} - \dfrac{a^2 + b^2 - 1}{a^2 + b^2 + 1} = \dfrac{2}{a^2 + b^2 + 1}$.

$$h_N(x, y, z) = h_N \left(\frac{2a}{1 + a^2 + b^2}, \frac{2b}{1 + a^2 + b^2}, \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \right)$$

$$\overset{\spadesuit}{=} \left(\frac{2a}{1 + a^2 + b^2}, \frac{2b}{1 + a^2 + b^2}, 0 \right) \times \frac{a^2 + b^2 + 1}{2} = (a, b, 0).$$

So $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is surjective, and hence it is bijective.

Stereographic Projection in Complex Coordinates

Recall that the map $h_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is given by

$$h_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right) \in \mathbb{C}.$$

If follow from the proof of the last proposition that the inverse $h_N^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$ is given by

$$h_N^{-1}(a+ib) = \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{a^2+b^2-1}{a^2+b^2+1} \right),$$

or in complex coordinates that

$$h_N^{-1}(w) = \left(\frac{2w}{1+|w|^2}, \frac{|w|^2-1}{|w|^2+1} \right),$$

where the first two coordinates in $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ are represented by a complex number instead.

Similarly, we can define the stereographic map from $\mathbb{S}^2 \setminus \{S\} \to \mathbb{C}$ with respect to the south pole $S = (0, 0, -1)$ instead of $N(0, 0, 1)$.

For any point $P(x, y, z)$ in the sphere $\mathbb{S}^2 \setminus \{S\}$, the ray $\overrightarrow{SP}$ from N to P meets the complex plane $\mathbb{C}$ at a point $h_S(P) = Q'$.
If we denote the coordinates of Q' by $c + id$, i.e. $Q(c, d, 0)$ in $\mathbb{R}^3$, then the map h_S can be described in terms coordinates of $P(x, y, z)$ by
$$h_S(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}, 0 \right).$$

In fact, we just need to replace $(0, 0, 1)$ by $(0, 0, -1)$ in $\heartsuit$, in which $\lambda = \frac{1}{1+z}$ as we have done previously. One can prove that

$h_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{C}$ is also bijective, and its inverse map is
$$h_S^{-1}(a, b, 0) = \left(\frac{2a}{1+a^2+b^2}, \frac{2a}{1+a^2+b^2}, \frac{1-a^2-b^2}{1+a^2+b^2} \right).$$

Consequently, we have $h_S^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{S\}$ and $h_S(0, 0, 1) = (0, 0, 0)$, so $h_S^{-1}(0, 0, 0) = (0, 0, 1)$.

For later use, $(1+\frac{a^2+b^2-1}{a^2+b^2+1})^{-1} = (\frac{2(a^2+b^2)}{a^2+b^2+1})^{-1} = \frac{a^2+b^2+1}{2(a^2+b^2)}\spadesuit$.
Recall $h_S(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}, 0\right) \quad \heartsuit$.

Theorem. The composition map $h_S \circ h_N^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is given by $h_S \circ h_N^{-1}(w) = \dfrac{1}{\overline{w}}$ for any $w \in \mathbb{C}$ in complex coordinates.

Proof. For any $w = a + ib \in \mathbb{C}$, then $h_S \circ h_N^{-1}(a+ib) = h_S \circ h_N^{-1}(a,b,0)$

$$= h_S\left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, \frac{a^2+b^2-1}{a^2+b^2+1}\right) \quad \heartsuit$$

$$\overset{\spadesuit}{=} \left(\frac{2a}{1+a^2+b^2}, \frac{2b}{1+a^2+b^2}, 0\right) \times \frac{a^2+b^2+1}{2(a^2+b^2)}$$

$$= \left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}, 0\right) = \frac{a+ib}{|a+ib|^2} = \frac{a+ib}{(a+ib)(a-ib)} = \frac{1}{a-ib}.$$

It is called inversion with respect to the center O in $\mathbb{C}$.

Another Inversion

We just proved that the composite map $h_S \circ h_N^{-1}(w) = \frac{1}{\overline{w}}$ $\heartsuit$.
How is other map $h_N^{-1} \circ h_S$? The answer is that both are the same.

Proposition. For any $w \in \mathbb{C} \setminus \{0\}$, $h_N \circ h_S^{-1}(w) = \dfrac{1}{\overline{w}}$,
where $\overline{w}$ is the complex conjugate of w.

Proof. For any $w \in \mathbb{C} \setminus \{0\}$, denote $w' = \frac{1}{\overline{w}}$. Then we have

$\spadesuit$ $\frac{1}{\overline{w'}} = \overline{(\frac{1}{w'})} = \overline{\frac{1}{(1/\overline{w})}} = \overline{(\overline{w})} = w$, hence $h_S \circ h_N^{-1}(w') \overset{\heartsuit}{=} \frac{1}{\overline{w'}} \overset{\spadesuit}{=} w$.

By composing with h_S^{-1}, we have $h_N^{-1}(w') = (h_S^{-1} \circ h_S) \circ h_N^{-1}(w')$
$= h_S^{-1} \circ (h_S \circ h_N^{-1})(w') = h_S^{-1}(h_S \circ h_N^{-1}(w')) = h_S^{-1}(w)$.
Then composing with h_S again,

$$\frac{1}{\overline{w}} = w' = h_N \circ h_N^{-1}(w') = h_N(h_N^{-1}(w')) = h_N(h_S^{-1}(w)) = h_N \circ h_S^{-1}(w).$$

Consequently, $h_N \circ h_S^{-1}(w) = \dfrac{1}{\overline{w}}$.

Translated Stereographic Projection

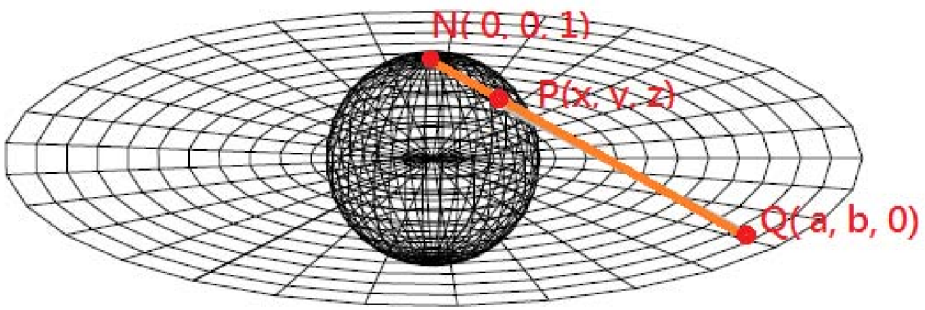
Remark. The map $h_S \circ h_N^{-1} = h_N \circ h_S^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a typical example in which one can construct mappings from certain subsets of $\mathbb{C}$ to itself by

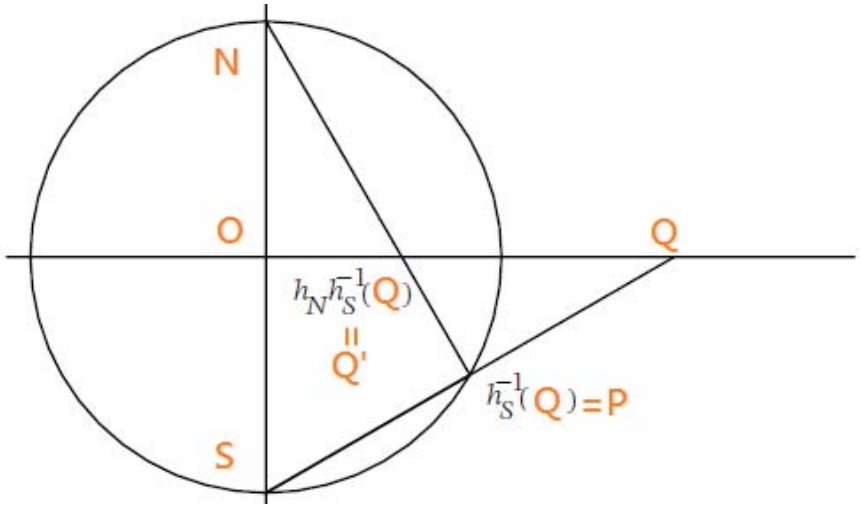
1. putting $\mathbb{C}$ into the punctured sphere $\mathbb{S}^2 \setminus \{N\}$ in a larger space $\mathbb{R}^3$, and
2. composite with certain (new) mapping in $\mathbb{R}^3$.

Question. How you define more mappings of this type?

Discussion. One simple way to define some new maps is to move the standard sphere $\mathbb{S}^2$ centered at $O = (0,0,0) = 0 \in \mathbb{C}$, to another point $(a,b,0) = a + ib \in \mathbb{C}$.
What is your guess on the formula of stereographic projection h_N?

Geometric Meaning of Inversion

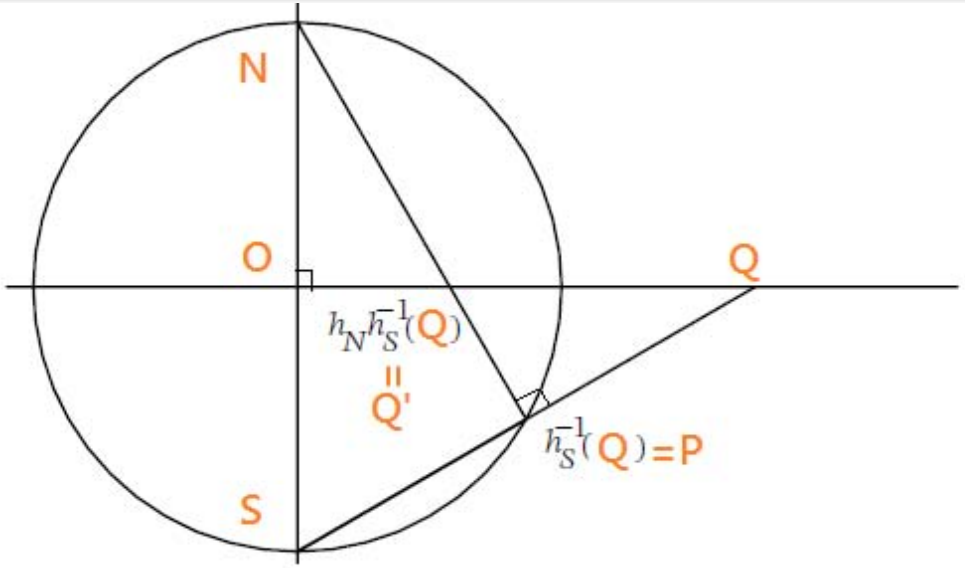


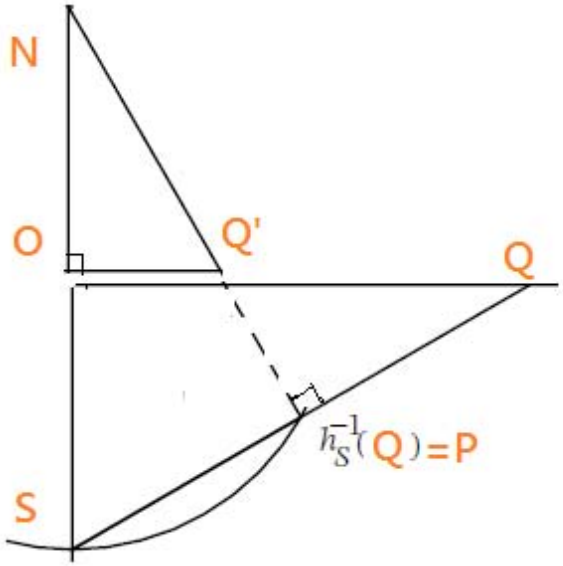


- For any $Q(a,b,0) \in \mathbb{C} \setminus \{0\}$, $P = h_S^{-1}(Q)$ is the intersection point of the ray $\overrightarrow{SQ}$ with the sphere $\mathbb{S}^2$ with center O.
- Similarly, $Q' = h_N \circ h_S^{-1}(Q)$ is the intersection point of the ray $\overrightarrow{NP}$ with the complex plane $\mathbb{C}$.
- So there is a plane π through O in $\mathbb{R}^3$ spanned by N, S and Q. The figure above on the right shows intersection of plane π and $\mathbb{S}^2$.
- The horizontal line OQ is the intersection of the plane π with the xy-plane (complex plane) in $\mathbb{R}^3$.
- The circle $\odot\, O$ is the intersection of the sphere and the plane.

Geometric Meaning of Inversion

Inversion Formula. $OQ \cdot OQ' = R^2$, where $R =$ radius of $\mathbb{S}^2$.





Proof. It suffices to show that $\triangle NOQ' \simeq \triangle QOS$.
First observe that NS is a diameter of circle $\odot O$, and $P = h_S^{-1}(Q)$ is on the circle, so $\angle NPS = 90^\circ$ i.e. $NP \perp SQ$.
So $\angle QNO = 90^\circ - \angle NQO = 90^\circ - \angle QQ'P = \angle PQQ' = \angle SQO$. It follows from AAA that $\triangle NOQ' \simeq \triangle QOS$. $\heartsuit$
Then $\dfrac{R}{OQ'} = \dfrac{NO}{OQ'} \overset{\heartsuit}{=} \dfrac{OQ}{OS} = \dfrac{OQ}{R}$, so $OQ \cdot OQ' = R^2$. $\square$

1. We will come back to this topics "inversion" in plane geometry!
2. However, one can define stereographic projection from $\mathbb{R}^n$ to $\mathbb{S}^n \setminus \{N\}$ with a similar geometric construction.
3. We have emphasized how the complex numbers work in the formula of inversion. Later, we will see more related to Möbius transformation, which will be very important in both Hyperbolic Geometry and even Number Theory.

Definition. Let A, B, C be subsets of X, we define

- $A = B$ if and only if the following holds:
  $x \in A$ if and only if $x \in B$.
- $A \subset B$ if and only if the following holds: If $x \in A$ then $x \in B$.
- $A = B$ if and only if the following holds: $A \subset B$ and $B \subset A$. i.e.
  $x \in A$ if and only if $x \in B$.
- $C = A \cup B$ if and only if $A \subset C$ or $B \subset C$.
- $C = A \cap B$ if and only if $A \subset C$ and $B \subset C$.
- $x \in A \setminus B$ if and only if $x \in A$ and $x \notin B$.

Remarks

1. Suppose that we know two statements P and Q hold, then we deduce P holds. Similarly we deduce Q holds as well.
2. If we know that P holds, then we know that P or Q will hold, no matter what Q is.
3. Suppose the following conditional statement holds:
   if P holds, then Q holds. We can rewrite this statement as: If Q does

Example. Let A, B, C, D be subsets of X, prove that

1. $A \cap B \subset A$ and $A \cap B \subset B$;
2. $A \subset A \cup B$ and $B \subset A \cup B$;
3. $A \setminus B \subset A$.

Proof.

1. $x \in A \cap B$ then $x \in A$ and $x \in B$, then $x \in A$. Thus we have proved that if $x \in A \cap B$, then $A \cap B \subset A$. Similarly, $A \cap B \subset B$.
2. Suppose that $x \in A$, then we have $x \in A$ or $x \in B$, i.e. $x \in A \cup B$, and hence $B \subset A \cup B$; Similarly, $A \subset A \cup B$.
3. If $x \in A \setminus B$, then $x \in A$ and $x \notin B$, so $x \in A$. Thus, $A \setminus B \subset A$.

Definition. Let $a \in X$ and $b \in Y$, define $(a, b) = \{ \{a\}, \{a, b\} \}$, which is called the ordered pair of a and b.

The following characterization of ordered pairs is very important.

Proposition. Prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Solution. Compare the elements a and b, we have the following two cases:

1. If $a = b$, then
   $\{\{c\}, \{c, d\}\} = (c, d) = (a, a) = \{ \{a\}, \{a, a\} \} = \{\{a\}\}$, so
   $\{c\} = \{a\} = \{c, d\}$, and hence $c = a = d$.
2. If $a \neq b$, then we have
   $\{\{c\}, \{c, d\}\} = (c, d) = (a, b) = \{ \{a\}, \{a, b\} \}, \quad \heartsuit$ so
   $\{c\} \neq \{a, b\}$; otherwise $a = c = b$, which is a contradiction. By $\heartsuit$, we have $\{c\} = \{a\}$, and $\{a, b\} = \{c, d\}$. It follows from the former equality that $a = c$, and then from the second equality that $b = d$.

Remark. You are not supposed to just remember the proof, as it is not related to the geometry, but instead its importance in coordinates, such as $(1, 0) \neq (0, 1)$. However, it is a very fundamental fact which illustrates the use of set theory.

Definition. Let X and Y be sets, then $X \times Y$ is a set of ordered pairs (x, y), where $x \in X$ and $y \in A$, i.e.

$$X \times Y := \{ (x, y) \mid x \in X \text{ and } y \in Y \}.$$

Definition. Let X and Y be sets, a function f from X to Y, denoted by $f : X \to Y$, is given by triple (X, Y, G_f), where $G_f \subset X \times Y$ satisfying the following condition: If (x, y_1) and $(x, y_2) \in G_f$, then $y_1 = y_2$.

Remarks.

1. The condition above is called the well-defined property of a function. Usually, If $(x, y) \in G_f$, then we denote by $y = f(x)$, i.e. $G_f = \{ (x, f(x)) \mid x \in X \}$, which is called the graph of f. The set X is called the domain of f.
2. Usually, we seldom mention the graph G_f of a function f, but it is part of the definition of a function f.

Let $f\colon X \to Y$ be a function from X to Y. For any $A \subset X$, define $f(B) = \{\ f(x) \in Y \mid x \in A\ \}$.

Remark. Let B be a subset of Y, then $f(A) \subset B$ if the following holds:

for any $x \in A$ we have $f(x) \in B$.

Definition. Let $f\colon X \to Y$ be a function from X to Y. For any $A \subset X$, define $f[A] = \{\ f(x) \in Y \mid x \in A\ \}$, which is called the (direct) image of A under f.

One can use the following characterization of the direct image:

$$y \in f[A] \text{ if and only if } \exists x \in A \text{ such that } y = f(x).$$

Definition. Let $f\colon X \to Y$ be a function from X to Y. For any $B \subset Y$, define $f^{-1}[B] = \{\ x \in X \mid f(x) \in B\ \}$, which is called the inverse image of B under f.

One can use the following characterization of the inverse image:

$$x \in f^{-1}[B] \text{ if and only if } f(x) \in B.$$

Obviously, $f^{-1}[B]$ is a subset of X, and $f[A]$ is a subset of Y, if $f\colon X \to Y$.

Definition. A function $f\colon X \to Y$ is called injective, if the following condition hold-
s:

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

Definition. A function $f\colon X \to Y$ is called surjective, if the (direct) image $f[X]$ of X under f is equal to Y, i.e. $f[X] = B$, or equivalently,

$$\text{for any } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

Definition. A function $f\colon X \to Y$ is called bijective, if$f$ is both injective and surjective.

Proposition. If $f\colon X \to Y$ is a bijective, then there exists a uniqe function $g\colon Y \to X$, called the inverse function of f such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.