6.1 Isometries of Euclidean space\(^1\)

6.1.1 Example of Isometries.

**Definition** A map \(f : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) is an isometry, if it preserves distances, i.e. for all \(x, y \in \mathbb{R}^3\), one has \(\|f(x) - f(y)\| = \|x - y\|\).

**Remark** Each reflection across a plane is an isometry, and we shall prove later the following important result:

**Theorem.** Every isometry is a composition of reflections.

**Example. Translation.** Let \(a \in \mathbb{R}^3\), define a map \(T = T_a : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) by \(T(x) = x + a\) for all \(x \in \mathbb{R}^3\), called translation of \(\mathbb{R}^3\) by \(a\). Prove that \(T\) is an isometry of \(\mathbb{R}^3\).

**Proof.** For any \(x, y \in \mathbb{R}^3\), we have \(T(x) - T(y) = (x + a) - (y - a) = x - y\), so \(\|T(x) - T(y)\| = \|x - y\|\), and hence \(T\) is an isometry.

**Example. Rotation.** Define \(Rot = Rot_{k, \theta} : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) by \(Rot(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)\), called the rotation of \(\mathbb{R}^3\) along the axis with direction \(k\) by an angle \(\theta\). Prove that \(Rot\) is a isometry of \(\mathbb{R}^3\).

**Proof.** For any \(x_1 = (x_1, y_1, z_1)\) and \(x_2 = (x_2, y_2, z_2)\), we have \(\|Rot(x_1) - Rot(x_2)\|^2 = \|((x_1 - x_2) \cos \theta - (y_1 - y_2) \sin \theta, (x_1 - x_2) \sin \theta + (y_1 - y_2) \cos \theta, z_1 - z_2)\|^2\) equal to \(\|((x_1 - x_2)^2 \cos^2 \theta - (y_1 - y_2)^2 \sin^2 \theta, (x_1 - x_2)^2 \sin^2 \theta + (y_1 - y_2)^2 \cos^2 \theta, z_1 - z_2)^2\) or \(\|((x_1 - x_2)^2 + (y_1 - y_2)^2) \cos^2 \theta + (z_1 - z_2)^2\) hence, \(Rot\) is an isometry of \(\mathbb{R}^3\).

**Example. Reflection across a plane through the origin.**

Let \(n\) be a unit vector in \(\mathbb{R}^3\). Let \(P = \{x \in \mathbb{R}^3 \mid x \cdot n = 0\}\) be the plane through 0 with normal direction \(n\). Define a map \(R : R_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) by \(R(x) = x - 2(x \cdot n)n\) for all \(x \in \mathbb{R}^3\), called reflection of \(\mathbb{R}^3\) across \(P\). Prove that \(R\) is an isometry of \(\mathbb{R}^3\).

**Proof.** For any \(x, y \in \mathbb{R}^3\), we have \(R(x) - R(y) = (x - 2(x \cdot n)n) - (y - 2(y \cdot n)n) = (x - y) - 2((x - y) \cdot n)n\), so \(\|R(x) - R(y)\|^2 = \|(x - y) - 2((x - y) \cdot n)n\|^2\) equal to \(\|(x - y)\|^2 - 4((x - y) \cdot n)((x - y) \cdot n) + 4((x - y) \cdot n)^2\|n\|^2\) or \(\|(x - y)\|^2 - 4((x - y) \cdot n)^2 + 4((x - y) \cdot n)^2\) hence \(R\) is an isometry of \(\mathbb{R}^3\).

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\(^1\)This note is temporary and the final version will be modified with more examples.
6.1.2 Projection from a point onto a plane

Let \( \Pi \) be a plane in \( \mathbb{R}^3 \) given by equation \( \mathbf{x} \cdot \mathbf{n} = d \) for some unit normal direction \( \mathbf{n} \in \mathbb{R}^3 \), i.e. \( ||\mathbf{n}|| = 1 \) and \( d \in \mathbb{R} \). For any \( \mathbf{x} \in \mathbb{R}^3 \), define \( \text{Proj}_\Pi(\mathbf{x}) \) to be the unique point \( \mathbf{y} \) in \( \Pi \) such that \( \text{dist}(\mathbf{x}, \Pi) = \text{dist}(\mathbf{x}, \mathbf{y}) \), called the orthogonal projection of \( \mathbf{x} \) onto the plane \( \Pi \).

**Proposition.** \( \text{Proj}_\Pi(\mathbf{x}) = \mathbf{x} + (d - \mathbf{n} \cdot \mathbf{x})\mathbf{n} \).

**Proof.** For any \( \mathbf{z} \in \Pi \), we have \( \mathbf{z} \cdot \mathbf{n} = d \) (\( * \)), and it follows from Cauchy-Schwarz inequality that \( ||\mathbf{z} - \mathbf{x}||^2 = ||\mathbf{n}||^2 \cdot ||\mathbf{z} - \mathbf{x}|| = ||\mathbf{n}||^2 \cdot ||\mathbf{z} - \mathbf{x}|| = ||\mathbf{n} \cdot (\mathbf{z} - \mathbf{x})||^2 \geq |\mathbf{n} \cdot (\mathbf{z} - \mathbf{x})|^2 \geq |d - \mathbf{n} \cdot \mathbf{x}|^2 \), and the equality holds if and only if \( \mathbf{n} \cdot (\mathbf{y} - \mathbf{x}) = 0 \) for some \( \mathbf{y} \in \Pi \). Hence \( \mathbf{y} - \mathbf{x} = t\mathbf{n} \) for some \( t \in \mathbb{R} \). As \( \mathbf{y} \) lies in the plane \( \Pi \), so it follows from \( ||\mathbf{n}||^2 = 1 \) that \( \mathbf{y} = \mathbf{x} + t\mathbf{n} \) satisfies \( d = \mathbf{y} \cdot \mathbf{n} = (\mathbf{x} + t\mathbf{n}) \cdot \mathbf{n} = \mathbf{x} \cdot \mathbf{n} + t \), i.e. \( t = d - \mathbf{n} \cdot \mathbf{x} \).

In particular, \( \text{Proj}_\Pi(\mathbf{x}) = \mathbf{y} + t\mathbf{n} = \mathbf{x} + (d - \mathbf{n} \cdot \mathbf{x})\mathbf{n} \).

**Definition.** Define a map \( R : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[
R(\mathbf{x}) = \mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n} = \text{Proj}_\Pi(\mathbf{x}) + (d - \mathbf{n} \cdot \mathbf{x})\mathbf{n},
\]

called the reflection (in \( \mathbb{R}^3 \)) across the plane of \( \Pi \). The reason for this name follows from the following

**Proposition.** (i) \( R(\mathbf{x}) = \mathbf{x} \) for any \( \mathbf{x} \in \Pi \).

(ii) \( R(R(\mathbf{x})) = \mathbf{x} \) for all \( \mathbf{x} \in \mathbb{R}^3 \).

(iii) \( \mathbf{n} \times (R(\mathbf{x}) - \mathbf{x}) = 0 \) for any \( \mathbf{x} \in \mathbb{R}^3 \), i.e. segment \( [\mathbf{x}, R(\mathbf{x})] \) is perpendicular to \( \Pi \).

(iv) \( \text{Proj}_\Pi(\mathbf{x}) = \frac{\mathbf{x} + R(\mathbf{x})}{2} \), for any \( \mathbf{x} \in \mathbb{R}^3 \), i.e. \( \text{Proj}_\Pi(\mathbf{x}) \) is the mid-point of \( \mathbf{x} \) and \( R(\mathbf{x}) \).

**Proof.** (i) For any \( \mathbf{x} \in \Pi \), one has \( \mathbf{x} \cdot \mathbf{n} = d \), so it follows from the definition of \( R \) that \( R(\mathbf{x}) = \mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n} = \mathbf{x} + 2(d - d)\mathbf{n} = \mathbf{x} \).

(ii) \( R(R(\mathbf{x})) = R(\mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n}) = (\mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n}) + 2(\mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n})\mathbf{n} = \mathbf{x} + 2(d - \mathbf{x} \cdot \mathbf{n})\mathbf{n} = \frac{\mathbf{x} + R(\mathbf{x})}{2} \).

(iii) It follows from \( R(\mathbf{x}) - \mathbf{x} = 2(d - \mathbf{n} \cdot \mathbf{x})\mathbf{n} \) that \( \mathbf{n} \times (R(\mathbf{x}) - \mathbf{x}) = 0 \).

(iv) For any \( \mathbf{x} \in \mathbb{R}^3 \), one has \( \text{Proj}_\Pi(\mathbf{x}) = \mathbf{x} + (d - \mathbf{n} \cdot \mathbf{x})\mathbf{n} = \frac{\mathbf{x} + R(\mathbf{x})}{2} . \)

**Remark.** (i) One can deduce that the following two (directed) segments
\( [\text{Proj}_\Pi(\mathbf{x}), R(\mathbf{x})] = -[\text{Proj}_\Pi(\mathbf{x}), \mathbf{x}] \), and both segments are perpendicular to \( \Pi \). In terms of picture, the segment \( [\text{Proj}_\Pi(\mathbf{x}), R(\mathbf{x})] \) is the mirror image of a stick \( [\text{Proj}_\Pi(\mathbf{x}), \mathbf{x}] \) along the mirror given by the plane \( \Pi \).

(ii) In fact, the definition of \( R \) works for any \( n \)-dimensional Euclidean space, i.e. \( \mathbb{R}^n \) with the standard Euclidean scalar (dot) product: for any \( \mathbf{x} = (x_1, \cdots, x_n) \) and \( \mathbf{y} = (y_1, \cdots, y_n) \), one has \( \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n \).

(iii) The map \( \text{Proj}_\Pi \) and \( R \) do not depend on the choice of \( \mathbf{n} \), though \( \mathbf{n} \) appears in the definition of these two maps, i.e. if one choose unit normal of \( \Pi \) to be \(-\mathbf{n} \) instead of \( \mathbf{n} \), both maps defined with \(-\mathbf{n} \) will agree with the ones defined with \( \mathbf{n} \).

**Definition.** A map \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) is called linear if \( T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}) \), for any \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) and \( \alpha, \beta \in \mathbb{R} \).

**Theorem.** A reflection \( R \) across a plane \( \Pi \) is a linear map if and only if \( \mathbf{0} \in \Pi \).

**Proof.** \(( \Rightarrow )\) Suppose that \( R \) is linear, then it follows from the definition of linear map that \( R(\mathbf{0}) = R(\mathbf{0} + \mathbf{0}) = R(\mathbf{0}) + R(\mathbf{0}) \), so by adding the vector \(-R(\mathbf{0}) \) to both sides, one has \( R(\mathbf{0}) = \mathbf{0} \). Then it follows from the definition of \( R \), one has \( \mathbf{0} = R(\mathbf{0}) = \mathbf{0} + (d - (0 \cdot \mathbf{n})\mathbf{n}) = d\mathbf{n} \). Taking the length of both sides and note that \( \mathbf{n} \) is a unit vector, so \( 0 = ||\mathbf{0}|| = ||d\mathbf{n}|| = |d| \cdot ||\mathbf{n}|| = |d|, \) so \( d = 0 \). Hence, \( \mathbf{0} \) satisfies the equation of \( \Pi \) as follows: \( d = 0 = 0 \cdot \mathbf{n} \). Hence \( \mathbf{0} \in \Pi \).

\((\Leftarrow)\) Suppose that \( \mathbf{0} \in \Pi \), then \( d = 0 = 0 \cdot \mathbf{n} = 0 \), so \( R(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n} \) for all \( \mathbf{x} \in \mathbb{R}^n \). One can then check that \( R(\alpha \mathbf{x} + \beta \mathbf{y}) = (\alpha \mathbf{x} + \beta \mathbf{y}) - 2((\alpha \mathbf{x} + \beta \mathbf{y}) \cdot \mathbf{n})\mathbf{n} = \alpha(\mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}) + \beta(\mathbf{y} - 2(\mathbf{y} \cdot \mathbf{n})\mathbf{n}) = \alpha R(\mathbf{x}) + \beta R(\mathbf{y}) \), for any \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) and \( \alpha, \beta \in \mathbb{R} \). Hence \( R \) is a linear map from \( \mathbb{R}^3 \) to itself. \( \square \)
6.1.2 Composition of Reflections.

Now consider two parallel planes, say \( \Pi_1 : \mathbf{x} \cdot \mathbf{n} = d_1 \) and \( \Pi_2 : \mathbf{x} \cdot \mathbf{n} = d_2 \), and let \( R_1 \) and \( R_2 \) denote the reflections across these planes.

**Proposition.** \( R_1 \circ R_2(x) = x + 2(d_1 - d_2) \mathbf{n} \) for any \( x \in \mathbb{R}^3 \).

**Proof.** First note that \( \mathbf{n} \) is a common unit normal of the parallel plane \( \Pi_1 \) and \( \Pi_2 \), we have 
\[
(\mathbf{x} + 2d_2 \mathbf{n}) - (\mathbf{x} + 2d_1 \mathbf{n}) = 2(d_2 - d_1) \mathbf{n}.
\]
Next, let \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be two non-parallel unit vectors in \( \mathbb{R}^3 \). Let \( \Pi_1 : \mathbf{x} \cdot \mathbf{n}_1 = d_1 \) and \( \Pi_2 : \mathbf{x} \cdot \mathbf{n}_2 = d_2 \) be any two planes with (non-parallel) normal directions \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) respectively. Let \( R_i \) be the reflection across the plane \( \Pi_i \) \( (i = 1, 2) \), It follows from \( \mathbf{n}_1 \times \mathbf{n}_2 \neq \mathbf{0} \) and question 2 of homework 4 that the intersection of these two planes \( \Pi_1 \) and \( \Pi_2 \) is a straight line, denoted by \( \ell = \Pi_1 \cap \Pi_2 \).

We consider the composition of reflections \( R_1 \circ R_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), which is a transform of \( \mathbb{R}^3 \). As \( R_i \) fixes each point of \( \Pi_i \) \( (i = 1, 2) \), i.e. \( R_i(x) = x \) for all \( x \in \Pi_i \), so each \( R_i(x) = x \) for \( x \in \Pi_1 \cap \Pi_2 = \ell \), so \( R_1 \circ R_2 \) fixes every point of \( \ell \). Hence we have

**Proposition.** (i) \( R_j \) fixes every point of \( \ell \). (ii) \( R_1 \circ R_2 \) also fixes every point of \( \ell \).

**Proposition.** Let \( \Pi \) be any plane perpendicular to \( \ell \), then \( R_j(\Pi) = \Pi \), for \( j = 1, 2 \).

**Proof.** Recall that the plane \( \Pi \) has normal vector \( \mathbf{n}_1 \), so the directional vector of the line is parallel to \( \mathbf{n}_1 \times \mathbf{n}_2 \), so \( \mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 \) is a directional vector of the line \( \ell \). For any plane \( \Pi \) that is perpendicular to \( \ell \), then an unit normal directional vector \( \mathbf{n} \) of \( \Pi \) is parallel to \( \mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 \). Then it follows from \( \mathbf{n} \cdot \mathbf{a} \neq 0 \) that the line \( \ell \) will meet the plane \( \Pi \) at a single point \( x_0 \). Then \( \Pi = \{ x \in \mathbb{R}^3 | (x - x_0) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0 \} \).

We need to prove that \( R_j(\Pi) = \Pi \) \( (j = 1, 2) \), i.e. for any \( x \in \Pi \), one has to show that \( (R_j(x) - x_0) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0 \) as follows. As \( x \in \Pi \), \( (x - x_0) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0 \) \( \ast \) and 
\[
R_j(x_0) = x_0 \text{ for } i = 1 \text{ and } 2, \text{ we have }
\]
\[
(R_j(x) - x_0) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0.
\]

Hence \( R_1 \circ R_2 = T_\mathbf{a} \). \( \square \)
Let $\theta = \angle(\Pi_1, \Pi_2) \in [0, \pi/2]$ be the (non-obtuse) angle between the planes $\Pi_1$ and $\Pi_2$. Recall that the line $\ell$ meets the plane $\Pi$ at $x_0$, and $R_i$ fixes every point of $\ell$, so $R_j(x_0) = x_0$ for $j = 1$ and $2$. Now we want to prove the following

Theorem. $R_1 \circ R_2$ is a rotation of the plane $\Pi$ about the point $x_0$ by angle $2\theta$.

**Proof.** Let $x \in \Pi$ be any plane in $\Pi$. And denoted by $y = R_2(x)$ and $z = R_1 \circ R_2(x)$ as shown in the figure above. Recall that $R_i(x_0) = x_0$, and reflections $R_1$ and $R_2$ are isometries, it follows that

$$
\|z - x_0\| = \|R_1 \circ R_2(x) - R_1 \circ R_2(x_0)\| = \|R_1(R_2(x)) - R_1(R_2(x_0))\| = \|R_2(x) - R_2(x_0)\| = \|x - x_0\|.
$$

It remains to show that $\angle(z - x_0, x - x_0) = 2\theta$. As $\Pi_1$ and $\Pi_2$ divide the plane $\Pi$ into 4 sectors, here we just work out one of these 4 cases as shown in the figure above, and the other 3 cases can be treated similarly with directed angles (c.f. Homework 6).

\[
\angle(z - x_0, x - x_0) = \angle(y - x_0, y - x_0) = \angle(y - x_0, x - x_0)
\]

\[
= 2\angle(\Pi_1, y - x_0) - 2\angle(\Pi_2, y - x_0)
\]

\[
= 2(\theta + \angle(\Pi_2, y - x_0)) - 2\angle(\Pi_1, y - x_0) = 2\theta.
\]

\square

Remarks. (i) The angles appeared in the last part of the proof above should be directed angles as $R_1 \circ R_2$ and $R_2 \circ R_1$ are not necessarily the equal, while the $\angle(\Pi_1, \Pi_2) = \angle(\Pi_2, \Pi_1)$.

(ii) We understand rotation about a point in a plane just like the clock arm in a grandfather old clock, and hence we can extend the idea of rotation of $\mathbb{R}^3$ about an axis, just like roasting a pignet in a fire. The idea of using a rotating axis to transform any point in $\mathbb{R}^3$ about this axis is exactly the composition of the two reflections across two non-parallel planes. For this reason, we give the working definition of the rotation in $\mathbb{R}^3$.

**Definition.** A rotation $R$ of $\mathbb{R}^3$ is the composition of 2 reflections of $\mathbb{R}^3$ across two distinct non-parallel planes $\Pi_1$ and $\Pi_2$. The line $\ell$ of intersection of the planes $\Pi_1$ and $\Pi_2$ is called the axis of the rotation $R$.

As reflections are isometries, so any rotation, as composition of two reflections, is also an isometry.

**Proposition.** Rotation of $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$.

**Example.** Recall $\text{Rot} = \text{Rot}_{k, \theta} : \mathbb{R}^3 \to \mathbb{R}^3$ defined in example 6.1.1.

\[
\text{Rot}(x, y, z) = (x \cos \theta - y \sin \theta, z \sin \theta + y \cos \theta, z).
\]

As the axis of rotation of Rot is $z$-axis, and Rot rotate $xy$-plane by angle $\theta$. So $R_1 = R_1 \circ R_2$ with $R_2$ is the reflection across the $xz$-plane $\Pi_2 : 0 = j \cdot x = y$, and $R_1$ is the reflection across the plane $\Pi_1 : 0 = (i \cos(\theta/2) - j \sin(\theta/2)) \cdot x = \cos(\theta/2)x - \sin(\theta/2)y$. 

\[
\text{Rot}(x, y, z) = (x \cos \theta - y \sin \theta, (i \cos(\theta/2) - j \sin(\theta/2)) \cdot x, z \sin \theta + y \cos \theta).
\]
Isometries as composition of reflections

Lemma (5). Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry with $f(0) = 0$.

Then for all $x, y \in \mathbb{R}^3$, we have (i) $\|f(x)\| = \|x\|$ and (ii) $f(x) \cdot f(y) = x \cdot y$ (that is, $f$ preserves norms and scalar products).

Proof. (i) For any $x \in \mathbb{R}^3$, it follows from property of isometry $f$ that $\|f(x)\| = \|f(x) - 0\| = \|f(x) - f(0)\| = \|x - 0\| = \|x\|$.

(ii) It follows from the polarization identity that for all $x, y \in \mathbb{R}^3$

$$x \cdot y = -\frac{1}{2} \cdot (-2x \cdot y) = -\frac{1}{2} (x^2 - y^2) \quad \text{and} \quad (f(x) \cdot f(y))^2 = x^2 - y^2 = 0,$$

Then we have $x \cdot f(x) = x \cdot y = 0$, i.e., $f(x) = x$ for all $x \in \mathbb{R}^3$.

Proof. Consider two following cases:

(i) $a = b$ : Let $\Pi$ be any plane through the points $0$ and $b$, then the reflection $R$ across the plane $\Pi$ satisfies $R(a) = a = b$ as $R$ fixes any point of $\Pi$.

(ii) $a \neq b$ : It follows that $\|a - b\| \neq 0$, then $n = \frac{a - b}{\|a - b\|}$ is a unit vector, and let $\Pi : x \cdot n = 0$ be the plane through $0$ with normal $n$, and $R : \mathbb{R}^3 \to \mathbb{R}^3$ by $R(x) = x - 2(x \cdot n)n$. Note that $\|a - b\|^2 = \|a\|^2 - 2a \cdot b + \|b\|^2 = 2\|a\|^2 - 2a \cdot b$. So $R(a) = a - 2(a \cdot n)n = a - 2(a \cdot \frac{a - b}{\|a - b\|}) \cdot \frac{a - b}{\|a - b\|} = a - \frac{2\|a\|^2 - 2a \cdot b}{\|a - b\|^2} (a - b) = (a - b) = b$. It follows from $R^2 = \text{id}$ that $R(b) = R(R(a)) = R^2(a) = a$.

Theorem (8). Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry of $\mathbb{R}^3$, then

(a) $f$ can be expressed as a composition of at most 4 reflections of $\mathbb{R}^3$.

(b) If $f(0) = 0$, then $f$ can be expressed as a composition of at most 3 reflections of $\mathbb{R}^3$, each of them fixing the origin 0.

Proof. (a) Step (1): If $f(0) = 0$, then proceed to Step (2) with $f_1 = f$ preserving 0. Otherwise, we may assume that $f(0) \neq 0$, let $a = f(0)/2$ be the midpoint of 0 and $f(0)$. And let $\Pi$ be the plane through $a$ with normal direction $n = \frac{a}{\|a\|} = \frac{f(0)}{\|f(0)\|}$. Then $f(0) = \|f(0)\|n \quad (\ast)$. The plane $\Pi$ is $x \cdot n = a \cdot n = \frac{f(0)}{\|f(0)\|}$.

Let $R_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection across the plane $\Pi$. Then $R_1 \circ f(0) = R_1(f(0)) = f(0) + 2 \frac{(f(0))}{\|f(0)\|}$.

As $R_1$ and $f$ are isometries, then $f_1 = R_1 \circ f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry preserving 0.

Step (2): If $f_1(k) = k$, proceed to Step (3) directly with $f_2 = f_1$ preserving both 0 and $k$. Otherwise, we may assume that $f_1(k) \neq k$. As $f_1$ is isometry, so $\|f_1(k)\| = \|k\|$. It follows from lemma (7) that there exists a reflection $R_2 : \mathbb{R}^3 \to \mathbb{R}^3$ across a plane through $0$ interchanging $f_1(k)$ and $k$, so $R_2(f_1(k)) = k$ and $R_2(0) = 0$. Define $f_2 = f_2 \circ f_1$, then $f_2$ is an isometry of $\mathbb{R}^3$ satisfying $f_2(k) = R_2 \circ f_1(k) = R_2(f_1(k)) = k$, and $f_2(0) = R_2 \circ f_1(0) = R_2(f_1(0)) = R_2(0) = 0$, i.e., preserving 0 and $k$.

Step (3): If $f_2(j) = j$, proceed to Step (4) directly with $f_3 = f_2$ preserving both 0, $k$ and $j$. Otherwise, we may assume that $f_2(j) \neq j$. As $f_2$ is isometry preserving 0, so it follows lemma (5) (ii) and $k = f_2(k)$ that $f_2(j - j) = f_2(j) \cdot f_2(k) - j \cdot k = j \cdot k - 0 = 0$. Let $n = \frac{f_2(j) - 1}{\|f_2(j) - 1\|}$, and $\Pi'$ be the plane through $0$ with unit normal $n$. So it follows from lemma (7) that the reflection $R_3$ of $\mathbb{R}^3$ across the plane $\Pi'$ interchanging $j$ and $f_2(j)$. As $k \cdot n = 0$, so $k, 0 \in \Pi'$, and hence $R_3$ preserves both 0 and $k$. Then $R_3 \circ f_2(j) = R_3(f_2(j)) = j$, $R_3 \circ f_2(k) = R_3(f_2(k)) = k$, and $R_3 \circ f_2(0) = R_3(f_2(0)) = R_3(0) = 0$. So the isometry $f_3 = R_3 \circ f_2$ preserves 0, $j$ and $k$.

Step (4): If $f_3(i) = i$, proceed to Step (4) directly with $f_4 = f_3$ preserving both 0, $k$, $j$ and $i$. Otherwise, we may assume that $f_3(i) \neq i$. As $f_3$ is isometry preserving 0, so it follows lemma (5) (ii), $k = f_3(k)$ and $j = f_3(j)$ that $f_3(i) \cdot k = f_3(i) \cdot f_3(k) = i \cdot k = 0$, and $f_3(i) \cdot j = f_3(i) \cdot f_3(j) = i \cdot j = 0$. Hence $f_3(i)$ is parallel to $j \times k = i$. Again by lemma (5), $\|f_3(i)\| = \|i\| = 1$, and hence $f_3(i) = \pm i$. As $f_3(i) \neq i$, so $f_3(i) = -i$. Define $R_4$ be the reflection of $\mathbb{R}^3$ across the plane through 0 with normal $i$. So we have $R_4 \circ f_3(0) = R_4(0) = 0$, $R_4f_3(k) = R_4(k) = -2(k \cdot i)i = k$, $R_4 \circ f_3(j) = R_4(j) = -2(j \cdot i)i = j$ and $R_4f_3(i) = R_4(-i) = -2(-i \cdot i)i = -i$. In any case, let $f_4 = R_4 \circ f_3$ preserving 0, $i, j$ and $k$, by Lemma (6), $f_4$ is the identity map of $\mathbb{R}^3$, i.e., $R_4 \circ \cdots \circ R_1 \circ f = \text{id}$, where $R_i$ is either reflection or identity map, as $R_0 \circ R_1 = \text{id}$ we have $f = (R_4 \circ \cdots \circ R_1)^{-1} = R_{1}^{-1} \cdots R_4^{-1} = R_1 \circ \cdots \circ R_4$, which is a composition of at most 4 reflections.

(b) As $f(0) = 0$, one can skip step (1) as in the proof of (a), and proceed to steps (2) − (4) as in proof of (a), hence $R_4 \circ \cdots \circ R_2 \circ f$ is identity where $R_i$ is either reflection or identity preserving 0, then (b) holds as $f = (R_4 \circ \cdots \circ R_2)^{-1} = R_2 \circ \cdots \circ R_4$. 


Representation of Isometries of $\mathbb{R}^3$

**Theorem.** The most general isometry $f : \mathbb{R}^3 \to \mathbb{R}^3$ is of the form $f(x) = A(x) + f(0)$, where $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, i.e. $A(x+y) = A(x) + A(y)$, and $A(\alpha x) = \alpha A(x)$ for all $x, y \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

**Proof.** Define $A : \mathbb{R}^3 \to \mathbb{R}^3$ by $A(x) = f(x) - f(0)$ for all $x \in \mathbb{R}^3$. Then $A = T_{-f(0)} \circ f$ is an isometry of $\mathbb{R}^3$, and $A(0) = f(0) - f(0) = 0$. It follows from Theorem 8 (b) that $A$ is composition of at most 3 reflections across 3 planes, each passing through 0. It suffices to show that (i) each reflection above is a linear map, and (ii) composition of linear maps is linear. For (i), Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection of $\mathbb{R}^3$ across a plane $\Pi : x \cdot n = 0$ with unit normal $n$ through 0, then $R(x) = x - 2(x \cdot n)n$ for all $x \in \mathbb{R}^3$. For any $x, y \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, we have

$$R(x+y) = (x+y) - 2((x+y) \cdot n)n = (x-2(x \cdot n)n) + (y-2(y \cdot n)n) = R(x) + R(y),$$

and $R(\alpha x) = (\alpha x) - 2((\alpha x) \cdot n)n = \alpha(x-2(x \cdot n)n) = \alpha R(x)$.

As for (ii), let $R$ and $S : \mathbb{R}^3 \to \mathbb{R}^3$ be two linear maps, then for any $x,y \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, we have $R \circ S(x+y) = R(S(x+y)) = R(S(x) + S(y)) = R(S(x)) + R(S(y))$.

**Remark.** In general, if $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map, then define vectors $v_i$ ($i = 1, 2, 3$) $v_1 = A(i) = (a_{11}, a_{21}, a_{31})$, $v_2 = A(j) = (a_{12}, a_{22}, a_{32})$, $v_3 = A(k) = (a_{13}, a_{23}, a_{33})$. We can rearrange the 9 entries in $v_i$ as a table, called matrix of $A$ in terms of the ordered basis $(i, j, k)$,

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$  

Then by the definition of linear map, we have

$$A(x, y, z) = A(xi + yj + zk) = A(xi) + A(yj) + A(zk) = xA(i) + yA(j) + zA(k) = xv_1 + yv_2 + zv_3 = (a_{11}x + a_{12}y + a_{13}z)i + (a_{21}x + a_{22}y + a_{23}z)j + (a_{31}x + a_{32}y + a_{33}z)k.$$  

If we write the entries in $(x, y, z)$ vertically downward instead of from left to right, it follows that one can recover the matrix product (defined by the as follows):

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}.$$  

**Theorem.** Let $f$ be an isometry of $\mathbb{R}^3$, then $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a bijective map, i.e. $f$ is (i) injective and (ii) surjective.

**Proof.** (i) If $f(x) = f(y)$, then $\|x - y\| = \|f(x) - f(y)\| = \|0\| = 0$, so $x = y$, so $f$ is injective.

(ii) It follows from the previous theorem that there exists a linear map $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(x) = Ax + f(0)$. For any $y \in \mathbb{R}^3$, want to show that there exists $x \in \mathbb{R}^3$ such that $f(x) = y$, which is equivalently to $Ax = y_0$ where $y_0 = y - f(0)$. As $f$ is injective, it follows that $A$ is also injective, so the homogeneous equation $Ax = 0$ (in matrix form) has unique solution $x \in \mathbb{R}^3$, it follows from Cramer’s rule that the matrix $[A]$ has a non-zero determinant, hence the non-homogeneous equation $Ax = y_0$ (in matrix form) has unique solution $x \in \mathbb{R}^3$, for any $y_0 \in \mathbb{R}^3$. Hence $A : \mathbb{R}^3 \to \mathbb{R}^3$ is surjective, and so is $f : \mathbb{R}^3 \to \mathbb{R}^3$. 


Recall the definition of group.

Definition. Let \( G \) be a non-empty set. A binary operation \( \circ \) on \( G \) is a map \( : G \times G \to G \) from \( G \times G \) to \( G \), denoted by \( (x, y) = x \circ y \), sometimes called the product of \( x \) and \( y \).

A set \( G \) with a binary operation, denoted by \((G, \circ)\), is called a group, if it satisfies the following 3 axioms:

(i) Associative: For any \( x, y, z \in G \), we have \((x \circ y) \circ z = x \circ (y \circ z)\);
(ii) Identity: There exists \( e \in G \) such that for \( e \circ x = x \circ e = x \) for all \( x \in G \);
(iii) Inverse: For all \( x \in G \), there exists \( y \in G \) such that \( x \circ y = e = y \circ x \).

Example. Let \( G = \mathbb{R} \setminus \{0\} \) be the set of all non-zero real (rational/complex) numbers. The binary operation \( \circ \) on \( G \) is given by usual multiplication, i.e. \( x \circ y = xy \) for all \( x, y \in \mathbb{R} \setminus \{0\} \).

Example. Let \( G \) be the set of rational/real/complex invertible \( n \times n \) matrices. Then \( G \) is a group under usual matrix multiplication.

Proof. It follows from \( \det(AB) = (\det A) \cdot (\det B) \) that the product of two invertible matrices is invertible, so \( G \) is closed under matrix multiplication. It remains to show that the above 3 axioms hold for \( G \):

(i) It follows from \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{kl} = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} A_{ij} B_{jk} C_{kl} \right) \)

\[ = \sum_{j=1}^{n} \left( \sum_{k=1}^{n} A_{ij} B_{jk} \right) C_{kl} \]

that matrix multiplication is associative.

(ii) The identity (as an element of the group \( G \)) under the binary operation of matrix multiplication is just the \( n \times n \) identity matrix \( I \) with 1 on the diagonal entries and 0 otherwise.

(iii) For any \( n \times n \) invertible matrix \( A \), we have a formula to write the matrix inverse \( A^{-1} \) of \( A \), such that \( A^{-1} \cdot A = I = A \cdot A^{-1} \). In fact, \( A^{-1} = \frac{\text{adj} A}{\det A} \), where the \( ij \)-entry of \( n \times n \) matrix \( \text{adj} A \) is equal to the \( ij \)-cofactor \( (-1)^{i+j} \det(B_{ij}) \), and the matrix \( B_{ij} \) is an \((n-1)\times(n-1)\) matrix obtained by deleting the \( i \)-row and \( j \)-column of \( A \). It follows from definition of binary operation in \( G \) that the inverse matrix \( A^{-1} \) of \( A \) is the inverse element of \( A \) (as element of \( G \)) that \( A \cdot A^{-1} = I = A^{-1} \cdot A \).

Theorem. The set \( \text{Isom}(\mathbb{R}^3) \) of isometries of \( \mathbb{R}^3 \) is a group with respect to composition. 
Proof. By the proposition of 6.1.1, composition of two isometries is an isometry, so the set \( \text{Isom}(\mathbb{R}^3) \) is closed under composition. Now we check the 3 axioms in the definition of a group:

(i) Associativity follows from the associative property of composition of maps.

(ii) It is obvious that the identity map \( \text{id} : \mathbb{R}^3 \to \mathbb{R}^3 \), defined by \( \text{id}(x) = x \) for all \( x \in \mathbb{R}^3 \), is the identity element of the set \( \text{Isom}(\mathbb{R}^3) \) under the binary operation of composition.

(iii) By Theorem (8), any isometry of \( \mathbb{R}^3 \) can be expressed as a composition of at most 4 reflections, and we know that the inverse of a reflection is itself, so the inverse \( f^{-1} \) of an isometry \( f \) of \( \mathbb{R}^3 \) (as a mapping from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \)) can be expressed as a composition of at most 4 reflections, i.e. the inverse \( f^{-1} \) is an isometry such that \( f \circ f^{-1} = f^{-1} \circ f = \text{id} \), so \( f^{-1} \) is the inverse of \( f \) under the binary operation of composition. \( \square \)
Definition. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of $\mathbb{R}^2$ across the line $\ell : y = x \tan \theta + c$. The line $\ell$ does not necessarily pass through the origin $(0,0)$. Then $R$ is given by $T^{-1} \circ R_0 \circ T$, where $T(x,y) = (x, y - c)$ and $R_0$ is the reflection of $\mathbb{R}^2$ across the line $y = x \tan \theta$. The result follows from the definition of reflection $R$ across the plane $\Pi : x \cdot n = 0$ that $R(x) = x$ for all $x \perp v$ and $R(v) = -v$. And hence the matrix of the linear transformation $R$ is equal to Householder matrix $Q_v$. It remains to show that $Q(x) = x$ for all $x \in \mathbb{R}^3$ perpendicular to $v$. Then in terms of column vector, we have
$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, so $x \perp v \iff x \cdot v = ax + by + cz = 0 \iff [a, b, c] \cdot [x, y, z] = 0$.

Hence, $Q_v(x) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} - \frac{2}{a^2 + b^2 + c^2} [a, b, c] \cdot [x, y, z] = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = x$. It follows from the definition of reflection $R$ across the plane $\Pi : x \cdot n = 0$ that $R(x) = x$ for all $x \perp v$ and $R(v) = -v$. And hence the matrix of the linear transformation $R$ is equal to Householder matrix $Q_v$. □

Remark. (ii) gives a way of implementing of the computation of reflection in terms of standard basis. One can search Householder QR Factorization for more details, which is fundamental in numerical linear algebra.